THE
RUDIMENTS OF MATHEMATICS;

DESIGNED FOR THE USE OF

STUDENTS AT THE UNIVERSITIES:

CONTAINING

AN INTRODUCTION TO ALGEBRA,
REMARKS ON THE FIRST SIX BOOKS OF EUCLID,
THE ELEMENTS OF PLANE TRIGONOMETRY:

BY W. LUDLAM,

LATE FELLOW OF ST. JOHN'S COLLEGE, CAMBRIDGE.

THE THIRD EDITION.

" Pure Mathematics do remedy and cure many defects in the wit, and faculties
" intellectual. For if the wit be dull, they sharpen it; if too wandering, they
" fix it; if too inherent in the sense, they abstract it. So that as Tennis is a game
" of no use in itself, but of great use in respect it maketh a quick eye, and a
" body ready to put itself into all positions; so, in the Mathematics, that use
" which is collateral, and intervenient; is no less worthy, than that which is
" principal and intended." BACON'S Advancement of Learning.

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M DCC XC.

PREFACE.

THE following tract was drawn up for the use of some young gentlemen in one of our Universities. It is certainly with good reason that young scholars in our Universities are put upon the Study of Mathematics. But surely it is not so much for their intrinsic value, as their use in qualifying young men to reason justly on more important subjects. Religion is of more importance (here as well as hereafter) than either Algebra or Geometry. The Laws of the moral world are a nobler subject, than the laws of the natural world, which respect only matter and motion. But it is of use for young men to exercise the reasoning faculty, first of all on mathematical subjects; because the ideas concerned (those of numbers especially) are very distinct, and false conclusions easily detected. Considering then the study of mathematics as intended to teach young persons to reason; it seems absolutely necessary that the Logical part should be strictly attended to. The first principles of every mathematical Science should be clearly laid down; the consequences derived from them, strictly demonstrated. In Geometry we have an excellent treatise of this kind: the Elements of Euclid, which perhaps is the best book of Logic extant: But in Algebra we are not so happy. If any where we might expect clear principles in so eminent a teacher as Professor Saunderson.

—But the truth is, the elementary part of his treatise was hastily drawn up. The friends of the Professor knowing he had given up the greatest part of his life to the art of teaching; supposed he had a complete treatise by him: He had only a set of Examples for learners, and dissertations on some curious and difficult subjects*. He had all the doctrinal part to draw up, when the book went to press; this too is vastly obscured by the affectation of Mystery. The Elementary part is also clogged with many things, not likely to be of use in the common course of Academical Studies.

Mac Laurin's Algebra is justly celebrated; and is indeed very useful to those who study the more difficult parts of Algebra, such as Equations of the higher orders, Infinite Series, &c.—But what reason is there for putting young scholars in our Universities, upon subjects so difficult, and of so little use in the study of Natural Philosophy? As much of Newton's *Principia*, as an undergraduate can, *or ought*, to read; may be understood without any acquaintance with Cubic Equations, much less those of higher orders†. Mac Laurin seems to hurry over the elementary part, to get at these difficult subjects, where he could dis-

* His Method of Fluxions, is only a set of examples.—The writer of this speaks from personal knowledge, being at that time a pupil of the Professor's and engaged with others, in reading some parts of the *Principia*.

† The difficulties in Newton's *Principia* do not, for the most part, arise from a want of skill in pure mathematics. The Commentators, now not a few, will supply an algebraic step that is wanting; or demonstrate a property of some figure, which Newton, hastening to greater matters, takes for granted. The difficulty is in the Logic of his Physical arguments; generally very concise. He thought all would see what he saw. And hence it is that we so often find converse propositions neither proved, nor an hint of the proof given. In one remarkable instance (Lib. I. Prop. XIV. Cor. 1.) this was objected to him by some learned foreigners: Therefore in the last Edition he has just hinted the proof; but so concisely; that to most, it needs further explanation.

play

play his abilities.—No man can get any credit by making an Horn-book for the babes in Mathematics; though it may be an useful work.

An apology may be necessary for dwelling so much on all occasions, in this tract, on the nature and use of the negative sign. It is however the subject of a quarto volume published by an excellent Mathematician; One who has explained some of the abstrusest parts of Algebra, on the clearest principles, and who is above valuing himself on a dexterity in the manœuvre of a parcel of symbols, such that no man can form an idea of what they represent*. It is with justice that Mr. Maseres complains, "It is by the introduction of "needless difficulties and mysteries into algebra, "(which for the most part take their rise from the "supposition of the existence of quantities *less than* "*nothing*) that the otherwise clear and elegant science

* Some very great algebraists talk of "extracting the *impossible* root of an *impossible* quantity, &c." These impossible quantities, have been made the subject of arithmetical operations; and their ratios, their sums, and their products have been computed. What is most strange; it is allowed on all hands, that just conclusions have been deduced in this way. In the Philosophical Transactions, vol. LXVIII. page 318, we find an ingenious enquiry (by Mr. *Playfair*) how this arithmetic of impossible quantities comes to be of any use? How the unintelligible operations of Symbols for imaginary quantities, should yet lead to real truths. Mr. *Playfair* supposes that they become useful, because they imply similar algebraic expressions in which no impossible quantities are involved, and which expressions belong to other subjects nearly related to those, from the investigation of which these impossible quantities arise.—There are it seems, certain quantities in Mathematics, between which there is a wonderful affinity: so that the algebraic expressions for the properties of the one; by a slight alteration, become the expressions for the properties of the other. This analogy is called by Mr. Cotes, the "Harmony between the Measures of Ratios and Angles." It is only in these singular cases that imaginary expressions lead to real truths; and the investigation is, at the bottom, only an argument from analogy, and has no claim to the full and clear evidence of Demonstration.

In all these cases both Mr. Maseres and Mr. Playfair, have shewn, how the same conclusions may be drawn, without introducing any *imaginary* quantities into the argument.

"of algebra, has been clouded and obſcured, and "rendered diſguſting to numbers of men of a juſt "taſte for reaſoning; who are apt to complain of it "and deſpiſe it on that account."

Philoſophical Tranſactions, vol. LXVIII. p. 947.

PROEM

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Containing the Doctrine of Vulgar Fractions.

1. THE following treatise, designed for students in our universities, was thought imperfect without a short summary of the doctrine of fractions. This is seldom taught in the schools of Arithmetic, but, instead of it, a variety of rules, useful perhaps in trade, but not at all wanted in the study of the sciences. What is here laid down is very short, yet it is hoped sufficient to instruct a learner in the sense of every rule, and to give him some view of the reason of it. The habit of performing the several operations readily can only be acquired by practice, and must be left to the learner's industry; but it is absolutely necessary, that every student in Algebra should be expert in these rules, as the rules for fractions in algebra are the very same as in arithmetic. For this reason they are not taught over again in the following treatise, but the learner is referred for them to the rules for fractions in arithmetic, which it is supposed he has learned already.

2. In common arithmetic, unity, or one, is the least number, which is the subject of the rules there delivered;

vered; but it is neceſſary on many occaſions to conſider arithmetical quantities leſs than one; to ſuppoſe unity to be broken into many equal parts, and to make a certain number of thoſe parts the object of conſideration. Such a number of equal parts is called a *Fraction*. Its notation is made by two numbers, one over the other under a ſtraight line. The number under that line is called the *Denominator*, and denominates how many equal parts the unit is ſuppoſed to be broken into. The number above that line is called the *Numerator*, and ſignifies how many of thoſe equal parts are reckoned up, or contained in that fraction. Thus one pound ſterling is ſuppoſed to be broken into 20 equal parts (called ſhillings), and in a crown are reckoned up or contained 5 of ſuch parts. If then we would conſider a crown as a fractional part of one pound, it muſt be written down thus, $\frac{5}{20}$, and it is read, *Five twentieths of a pound*. In like manner we write down three farthings, thus, $\frac{3}{4}$; ſignifying that the penny is ſuppoſed to be broken into 4 equal parts, and that we take 3 of them in that fraction, and it is read, *Three fourths of a penny*.

3. It is evident that if we take into the value of a fraction a greater number of parts than the whole unit was originally broken into, we take more than the whole; ſo, $\frac{21}{20}$ (*Twenty one twentieths*) of a pound is more than a pound; it is an intire pound, and one twentieth part of a pound over. Such a fraction, not being a part of a pound, but more than the whole, is called an *improper fraction*. In the ſame manner, if we take a number of parts, exactly equal to that number into which the whole unit was originally broken, we take the whole; and this alſo is called an improper fraction. Thus $\frac{20}{20}$ of a pound is an intire pound. In general every fraction, whoſe numerator and denominator are equal, is equivalent to unity.

4. Hence if an improper fraction be propoſed, a queſtion will ariſe, " What whole number, or what " whole

"whole number with ſome fraction beſides (called a "*mixt number*) is equivalent to this improper fraction?" The rule is, *Divide the numerator by the denominator, and the quotient is the integral part of the mixt number; make a fraction, whoſe numerator is the remainder (of this diviſion), and denominator the diviſor, and that will be the fractional part.* Thus $\frac{63}{20}$ is equivalent, or equal to $3\frac{3}{20}$; and $\frac{63}{7}$ is the ſame in value as 9 integral, only expreſſed in the form of a fraction. In other words, to divide any integer into 7 equal parts, and to take 63 ſuch parts, is juſt the ſame as to take 9 integers. To divide a lottery ticket into 7 ſhares, and to take 63 ſuch ſhares, is in fact to take 9 whole tickets. Hence the reaſon of the rule will appear; for to divide 63 by 7 is to enquire how many ſevens there are in 63, and what remains over. As often then as the denominator is found in the numerator, ſo many integers or units there are in the value of that fraction; what remains over are ſo many of the original fractional parts.

5. Another queſtion ariſes out of this ſubject, the converſe of the former. A mixt number being propoſed, to find what improper fraction expreſſes the ſame value. The rule is, *Multiply the integral part by the denominator of the fractional part, and to the product add the numerator of the fractional part, and this ſum ſhall be the numerator of the improper fraction ſought; under it ſubſcribe the denominator of the fractional part of the mixt number, for the denominator of the improper fraction ſought.* Thus $3\frac{3}{20}$ is equivalent to $\frac{63}{20}$. The truth of this will appear from conſidering, that every integer or unit is equal in value to ſo many parts as it is broken into; that is, to ſo many fractional parts as are expreſſed by the denominator of the fraction.

6. Every whole number may be conſidered as an improper fraction, whoſe numerator is that number, and whoſe denominator is unity, or 1. Thus $\frac{20}{1}$ means 20 integers, or 20 units, or 20.

7. We ſhall now lay down the leading principles on which all the rules of fractions are founded, and which alſo are of great uſe on other occaſions, and therefore ſhall illuſtrate them by examples.

8. The value of every whole, made up of a number of equal parts, depends both on the magnitude and number of thoſe parts. If the number of parts in two inſtances are the ſame, but the magnitude of each part in one caſe is double, triple, &c. the magnitude of each part in the other; then the value of the whole in one caſe is double, triple, &c. the value of the whole in the other. Thus the value of ten ſhillings is double the value of ten ſix-pences.

9. Again, If the magnitude of each equal part in two inſtances is the ſame, but the number of parts in one caſe is double, triple, &c. the number of parts in the other; then the value of the whole in one caſe is double, triple, &c. the value of the whole in the other. Thus the value of ten ſix-pences is double the value of five ſix-pences.

This may be otherwiſe expreſſed thus:

10. If the magnitude of each equal part be increaſed while the number of parts remain the ſame, the value of the whole will be increaſed proportionably; and contrariwiſe.

11. If the number of equal parts be increaſed, while the magnitude of each part remains the ſame, the value of the whole will be increaſed proportionably; and contrariwiſe.

12. If any whole, or integer, is to be divided into a number of equal parts, then the value of one of thoſe parts depends partly on the magnitude of the whole to be ſo divided, and partly on the number of parts into which it is to be divided.

13. If the number of equal parts into which the whole is to be divided in two inſtances is the ſame, but the magnitude of the whole in one caſe is double, triple, &c. the magnitude of the whole in another; then the value of each part in one caſe is double,

triple,

triple, &c. the value of each part in the other. Thus a ſixteenth ſhare of 20 pounds is double a ſixteenth ſhare of 10 pounds.

14. If the whole to be divided is in two inſtances the ſame, but the number of equal parts into which that whole is to be divided is in one caſe double, triple, &c. the number of parts into which that whole is to be divided in another; then the value of each part in one caſe is only one half, one third, &c. of the value of each part in the other, increaſing the number of parts, decreaſing the value of each part; and contrariwiſe. Thus a ſixteenth ſhare of a twenty pound prize is only half (in value) of an eighth ſhare of the ſame prize.

This may be otherwiſe expreſſed thus:

15. If the magnitude of the whole be increaſed, while the number of (equal) parts remain the ſame, the value of each part will be increaſed proportionably; and contrariwiſe.

16. If the number of equal parts into which the whole is divided be *increaſed*, while the magnitude of the whole remains the ſame, the value of each part will be *decreaſed* proportionably; and contrariwiſe.

17. Theſe principles cannot be proved, they want no proof. No one who underſtands their meaning will doubt of their truth. Such truths as theſe (being ſelf-evident) are called *Axioms*.

18. Every fraction is equivalent to the quotient of the numerator, divided by the denominator. In other words, it is the ſame thing whether we take a ſingle integer, and divide it into ſo many parts, 4 for inſtance, and take 3 of thoſe parts, or whether we take 3 ſuch integers, divide them into 4 parts, and take only one part. For, in the former caſe, 3 of thoſe parts is 3 times as much as 1 of the ſame parts, becauſe their magnitude is the ſame, by par. 9; and, in the latter caſe, the one part taken is alſo 3 times as much as one of the parts in the former caſe, becauſe the magnitude of the whole to be divided in the lat-

ter caſe, viz. 3 integers, is 3 times as much as the whole to be divided in the former caſe; viz. one integer, by par. 13. Both theſe quantities therefore are equal; being each 3 times a quarter part of one integer. And thus 3 quarters of one pound are equal to one quarter of 3 pounds. Both are juſt 3 times a quarter part of one pound.

19. In diviſion in common arithmetic, the quotient is often incomplete; becauſe there is a remainder which cannot be divided by the diviſor; but admitting fractions, it may be always made complete thus. Make a fraction whoſe numerator is that remainder, and denominator the diviſor, and this ſubjoined to the integral part of the quotient makes it perfect. Thus divide 23 by 4, the quotient is 5, the remainder is 3, which ought alſo to be divided by 4: but we have juſt ſhown, that $\frac{1}{4}$th part of 3 is the ſame as $\frac{3}{4}$ths of one; therefore the quotient of (the remainder) 3 divided by 4 is the fraction $\frac{3}{4}$, and thus the complete quotient will be $5\frac{3}{4}$.

20. *To eſtimate the fractional parts of an integer in parts of a leſſer denomination*; *and* vice verſa. To find for inſtance the value of $\frac{5}{6}$ of a pound. It has been ſhown that $\frac{5}{6}$ of one pound is the ſame as $\frac{1}{6}$ of 5 pounds; bring therefore 5 pounds into ſhillings by multiplying 5 by 20, and it makes 100 ſhillings, divide this by 6, and you will have (by par. 19) $16\frac{4}{6}$ for the number of ſhillings, equal to $\frac{5}{6}$ of a pound. In the ſame way $\frac{4}{6}$ of a ſhilling will be found equal to 8 pence; and ſo $\frac{5}{6}$ of a pound is in value 16 ſhillings and 8 pence.

21. For the reverſe, to find what fractional part of a pound, 16 ſhillings and 8 pence is; reduce the whole to pence, and it makes 200 pence. Now 240 pence make a pound. Therefore if one pound be divided into 240 parts, 200 of thoſe parts are equal to 16 ſhillings and 8 pence, which is therefore $\frac{200}{240}$ of a pound. We ſhall afterwards ſhow, that the fraction $\frac{200}{240}$, and the fraction $\frac{5}{6}$, are of the ſame value.

22. All the operations of fractions depend on two maxims. *First, If the numerator of a fraction be encreased while the denominator continues the same, the value of the fraction will be encreased proportionably; and contrariwise.* For while the denominator continues the same, the magnitude of the parts into which the unit is divided continues the same, therefore the value of the fraction will be proportionable to the numerator, by par. 11. But, *secondly, If the denominator be encreased in any proportion, while the numerator continues the same, the value of the fraction will be diminished in a contrary proportion; and contrariwise.* For the number of parts contained in the fraction continues the same; but the value of each part is changed in a contrary proportion to the denominator, by par. 16. Therefore the value of the whole fraction is changed in the same contrary proportion by par. 10.

23. From these two principles it follows, that if the numerator and denominator of a fraction be both multiplied, or both divided by the same number, the value of the fraction will not be affected thereby.

24 A. Hence every fraction is capable of being expressed by an infinite variety of numbers; for there are an infinite variety of multiplicators whereby the numerator and denominator of a fraction being each multiplied, the terms in which the fraction is expressed will be changed, but not its value. Thus the fractions $\frac{3}{4}$, $\frac{6}{8}$, $\frac{12}{16}$, &c. are all of the same value, though expressed in different terms, or by different numbers, the numerator and denominator of each, successively, being doubled.

24 B. As a fraction may be thus expressed in a variety of terms, a question will arise, what are the lowest terms, or least numbers, by which it can be expressed? It is evident from what has been said, that if any number can be found which will exactly divide both the numerator and denominator without leaving a remainder, then if such division be actually made, the two quotients will give the numerator and

denominator of another fraction equal to that proposed, but expressed in lower terms. Thus if the fraction $\frac{200}{240}$ was proposed, both the numerator and denominator can be divided by 10; and by such a division the fraction will be reduced to $\frac{20}{24}$. This again (by a division by 2) will be reduced to $\frac{10}{12}$; and this (by another division by 2) will be reduced to $\frac{5}{6}$.

25. Any number that will thus divide both numerator and denominator, without leaving a remainder, is called their *common divisor.*

26. The difficulty is to find such a common divisor; the following observations may be of service.

The number 2 will divide all even numbers, that is, all numbers ending with 2, 4, 6, 8, or a cypher.

The number 10 will divide all numbers ending with a cypher.

The number 5 will divide all numbers ending with 5, or with a cypher.

If none of the above numbers will divide both numerator and denominator of the proposed fraction, then try 3, 7, 11, 13, 17, 19, &c. successively, dividing both numerator and denominator by each of them as often as it is possible. Thus $\frac{378}{504}$, after a continual division by 2, 3, 3, 7, successively, is reduced to $\frac{3}{4}$. The fraction $\frac{375}{1125}$, after a continual division by 3, 5, 5, 5, is reduced to $\frac{1}{3}$. The fraction $\frac{7350}{8820}$, after a continual division by 10, 3, 7, 7, is reduced to $\frac{5}{6}$, therefore is of the same value with $\frac{200}{240}$, before proposed, par. 24 B.

27. Three or more numbers are said to be multiplied together *continually*, when the first is multiplied by the second, and that product is multiplied by the third, that product by the fourth, and so on.

28. In such a case it matters not in what order the numbers be taken, the product of the continual multiplication of all of them will be the same. Thus, let the several multipliers be 2, 3, 4, 5; whether we take them in that order, or in any other, as 4, 2, 5, 3, or

or 5, 4, 3, 2, the product of the continual multiplication of all of them will always be 120.

29. *The reduction of fractions having different denominators to others of the same value, having one common denominator.*

Rule. Put down the fractions to be reduced in any order. Then multiply the numerator of the first fraction *into* (that is, *by*) all the denominators but its own, for a new numerator to be set underneath that fraction, and so proceed to every one. Lastly, multiply all the denominators together for a common denominator to be set under each numerator before found.

Let the fractions be - $\frac{1}{2}$ $\frac{3}{4}$ $\frac{5}{6}$ $\frac{7}{8}$.
The new fractions will be $\frac{192}{384}$ $\frac{288}{384}$ $\frac{320}{384}$ $\frac{336}{384}$.

30. Whoever considers the work attentively, will see that both the numerator and denominator of every fraction is equally multiplied. Both are multiplied by the denominators of all the other fractions *continually* (par. 27). Consequently, their value is not altered (par. 23); that is, the value of the new fractions so found is the same with that of the old ones respectively. This will fully appear by reducing each of the new fractions successively to their lowest terms. Thus $\frac{192}{384}$, after a continual division by 2, six several times, and then by 3, will be reduced to $\frac{1}{2}$. The fraction $\frac{288}{384}$, after a continual division by 2, five several times, and then by 3, will be reduced to $\frac{3}{4}$. The fraction $\frac{320}{384}$, after a continual division by 2, six several times, will be reduced to $\frac{5}{6}$. And, lastly, the fraction $\frac{336}{384}$, after a continual division by 2, four several times, and then by 3, will be reduced to $\frac{7}{8}$.

31. If two or more fractions are proposed, having different numerators and also different denominators, it is impossible in that state to compare their value, or tell which is the greatest. But let them be reduced to other fractions of the same value and having one common denominator, then the comparison is easy and evident. Thus, should it be asked, which is the greatest

greatest $\frac{2}{3}$ or $\frac{3}{4}$, it would not be easy to determine this; the number of parts in the latter case is the greater, for 3 is more than 2; but the magnitude of each part is less, for a fourth part is less than a third part. On which side lies the advantage? Now let these fractions be reduced to two others of the same value respectively, but having one common denominator, to wit, $\frac{8}{12}$ and $\frac{9}{12}$, then it is evident that the latter is the greatest by $\frac{1}{12}$; for the same reason that 9 pennies are greater than 8 pennies by one penny; for a penny is the 12th part of a shilling.

Addition of Fractions.

32. When any number of fractions are reduced to others having the same denominator, we may then *(and not before)* find the sum of them all, by adding their several numerators together, and under that sum subscribing the common denominator. Thus, in the foregoing example, suppose it was required to find the sum of all the fractions $\frac{1}{2}$, $\frac{3}{4}$, $\frac{5}{6}$, $\frac{7}{8}$, these reduced to one denomination will be $\frac{192}{384}$, $\frac{288}{384}$, $\frac{320}{384}$, $\frac{336}{384}$. Now their several numerators, viz. 192, 288, 320, 336, added together make 1136, underneath this subscribe the common denominator, and the sum of all these four fractions will be the fraction $\frac{1136}{384}$: and this fraction by a continual division of both numerator and denominator by 2, four several times, will be brought to $\frac{71}{24}$. But $\frac{71}{24}$ is an improper fraction, which therefore must be brought to a mixt number by par. 4, to wit, to $2\frac{23}{24}$. So at last the sum of the four original fractions is found to be $2\frac{23}{24}$, wanting $\frac{1}{24}$ of an unit of the whole number 3.

33. After all operations, whether addition, subtraction, multiplication, or division, the fraction resulting should be brought to its lowest terms, and if an improper fraction to an equivalent mixt number.

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If whole numbers are to be added to fractions, it will be beſt to reduce ſuch whole numbers to the form of a fraction by par. 6, and then treat them as ſuch.

Subtraction of Fractions.

34. When two fractions are propoſed, let them be reduced to two others of the ſame value reſpectively, but having the ſame denominator, by par. 29, then it will appear which is the greateſt, viz. that which has the greateſt numerator. Subtract the leſs numerator from the greater, put down the difference as the numerator, and the common denominator as the denominator of a fraction, which is the difference between the two propoſed fractions.

35. Let the two fractions be $\frac{5}{12}$ and $\frac{3}{8}$; theſe reduced to two others of the ſame value, having one common denominator, are $\frac{40}{96}$ and $\frac{36}{96}$, wherefore $\frac{3}{8}$ is leſs than $\frac{5}{12}$, and their difference is $\frac{4}{96}$; this, by two ſucceſſive diviſions of both numerator and denominator by 2, is brought to $\frac{1}{24}$; ſo that at laſt the difference between the two fractions $\frac{5}{12}$ and $\frac{3}{8}$ is found to be $\frac{1}{24}$.

Again: let the fractions be $\frac{7}{12}$ and $\frac{11}{15}$; theſe reduced to the ſame denomination, are $\frac{105}{180}$ and $\frac{132}{180}$, wherefore the latter ($\frac{11}{15}$) is the greater, and their difference is $\frac{27}{180}$; this, by two diviſions by 3, is reduced to $\frac{3}{20}$; therefore $\frac{11}{15}$ is greater than $\frac{7}{12}$, and their difference is $\frac{3}{20}$.

Laſtly, let the numbers be $5\frac{7}{8}$ and $9\frac{1}{6}$. Theſe reduced to improper fractions by par. 5, are $\frac{47}{8}$ and $\frac{55}{6}$. Theſe reduced to one denomination are $\frac{282}{48}$ and $\frac{440}{48}$, and their difference is $\frac{158}{48}$. This reduced to the loweſt terms is $\frac{79}{24}$. Reduced to a mixt number is $3\frac{7}{24}$: the difference between the propoſed numbers.

Mul-

Multiplication and Division of fractions.

Before we lay down the rules, it may be proper to make ſome obſervations on the nature of multiplication and diviſion.

36. When the multiplicand continues the ſame, the value of the product will depend on the multiplier, as was ſaid par. 11. The greater or leſs the multiplier, the greater or leſs is the product, and that in the ſame proportion. Let 12 be multiplied by 4, and the product is 48; but if the multiplier is only half as much, viz. 2, the product will be only half; that is, 24. Again, let the multiplier be only half of 2, or 1, and the product is only half 24, or 12. And for the ſame reaſon, if the multiplier is only half of 1, or $\frac{1}{2}$, the product muſt be only half of 12, or 6. We do not here take 12 once, or one whole time, but only half a time; in other words, we take half of 12, which is 6. And whenever the multiplier is leſs than one, or ſome fractional part of one, the product will be leſs than the multiplicand. This may ſeem ſtrange to thoſe who are uſed to whole numbers only; but the contradiction is verbal only, conſiſting in an improper application of the term *multiply*. The term was proper enough, when applied to whole numbers only; and the old term is continued, now Arithmetic is extended to broken numbers, where the multiplier may be leſs than an unit; the idea has outgrown the name, a caſe very frequent in other ſciences.

37. Analogous to this, the quotient in diviſion, when whole numbers only are concerned, is always leſs than the dividend: but in the arithmetic of fractions it is not always ſo. In diviſion, we enquire how often the diviſor is found in the dividend. If the diviſor be one, it will be found as often as there are units in the dividend, and thus the dividend and quotient will be equal. But if the diviſor be leſs than unity, it

it muſt be found oftener than there are units in the dividend. So if 12 be divided by 1, the quotient is 12; but if 12 be divided by the fraction $\frac{1}{2}$, the quotient will be 24. The fraction $\frac{1}{2}$ will be found in 12 24 times, becauſe it is found twice in every unit.

Multiplication of Fractions, when the Multiplier is a Whole Number.

38. This will be rightly done, if we multiply the numerator of the fraction by the multiplier, for a new numerator, and keep the denominator the ſame; as will appear from conſidering the maxim in the former part of par. 22.

39. But this may alſo be done, by keeping the numerator the ſame, and dividing the denominator by the multiplier (when it can be done without leaving a remainder), and taking that quotient for a new denominator. This will appear from conſidering the maxim in the latter part of par. 22.

40. Thus $\frac{3}{4}$ may be multiplied by 2, either by doubling the numerator, and keeping the denominator the ſame, in which caſe the product will be $\frac{6}{4}$, or by keeping the numerator the ſame, and halving the denominator, in which caſe the product will be $\frac{3}{2}$. In the firſt method we ſay, that 6 quarters is double of 3 quarters. In the latter method we ſay, that 3 halves is alſo double of 3 quarters; both which are true. The former method is always practicable, the latter not always, but when it can be done it gives the product in lower terms.

Diviſion of Fractions, when the Diviſor is a Whole Number.

41. This will be rightly done, if we divide the numerator of the fraction by the diviſor (when ſuch diviſion can be made without leaving a remainder) and

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taking the quotient for a new numerator, keeping the denominator the ſame.

42. But this may alſo be done by keeping the numerator the ſame, and multiplying the denominator by the diviſor, and taking the product for a new denominator. Both theſe rules will appear to be juſt, from the two maxims in par. 22, aforeſaid.

43. Thus $\frac{4}{5}$ may be divided by 2, either by halving the numerator, and keeping the denominator the ſame, in which caſe the quotient will be $\frac{2}{5}$, or by keeping the numerator the ſame, and doubling the denominator, in which caſe the quotient will be $\frac{4}{10}$. That $\frac{4}{10}$ is of the ſame value with $\frac{2}{5}$ will appear from par. 24 B.

The latter method is always practicable; the former not always; but where it can be done, it gives the quotient in lower terms.

Multiplication, where both Multiplier and Multiplicand are Fractions.

44. An example may ſerve both to ſhew the rule and explain the reaſon of it.

Let it be required to multiply $\frac{3}{5}$ by $\frac{2}{3}$. Suppoſe now that inſtead of $\frac{2}{3}$ the multiplier was the numerator only of this fraction, viz. 2, conſidered as integral; then we ſhould find the product to be $\frac{6}{5}$ by par. 38: but we have here ſuppoſed the multiplier to be 3 times as great as it really is, for the multiplier is only a third part of 2, by par. 18, therefore our product will be 3 times too great, and a third part of it will be the true product. Divide therefore $\frac{6}{5}$ by 3, according to the method of par. 42, becauſe it is always practicable, and we have the true product $\frac{6}{15}$, or in lower terms $\frac{2}{5}$.

Hence this *rule* for multiplication:

45. *Multiply the numerator of the multiplier and multiplicand together for the numerator, and their denominators together for the denominator of the product.*

Diviſion,

Division, where both the Divisor and Dividend are Fractions.

46. We must argue just in the same way, to find the rule for division. Thus, let it be required to divide $\frac{3}{5}$ by $\frac{2}{3}$. Suppose now the divisor was the numerator only of the fraction $\frac{2}{3}$, that is, suppose the divisor was the integral number 2, we should then find the quotient to be $\frac{3}{10}$, making the division according to the method of par. 42, because it is always practicable. But we have here supposed the divisor to be 3 times as great as it really is (by par. 18), therefore, the quotient so found will be only a third part of what it should be. Multiply therefore $\frac{3}{10}$ by 3, according to par. 38, and we have the true quotient $\frac{9}{10}$.

Hence this *rule* for division:

47. *Invert the terms of the divisor* (not of the dividend), *then follow the rule for multiplication, par.* 45, *viz. Multiply the numerators together for a new numerator, and denominators together for a new denominator.*

48. By *inverting* the terms of a fraction is meant making another fraction such, that the numerator of the former shall be the denominator of the latter, and the denominator of the former the numerator of the latter. Such fractions are called reciprocals. So $\frac{2}{3}$ and $\frac{3}{2}$ are reciprocals.

49. From the rule before laid down it appears, that to divide one fraction by another, or to multiply the dividend by the reciprocal of the divisor, is the same thing. To divide $\frac{3}{5}$ by $\frac{2}{3}$ is the same as to multiply $\frac{3}{5}$ by $\frac{3}{2}$. So likewise to divide the number 12 by 2, is the same as to multiply 12 by $\frac{1}{2}$, for the whole number 12, may be considered as the improper fraction $\frac{12}{1}$ by par. 6, and the number 2 as $\frac{2}{1}$, and to divide $\frac{12}{1}$ by $\frac{2}{1}$ is the same as to multiply $\frac{12}{1}$ by $\frac{1}{2}$, and so the rule of division may be always changed into that of multiplication.

A question or two to exercise the rules of fractions.

Four persons *A*, *B*, *C*, *D*, share a Lottery ticket between them. *A* has $\frac{1}{3}$, *B* $\frac{1}{4}$, *C* $\frac{1}{5}$, and *D* the remainder; What share of the whole has *D*, and who has the least share?

In Wing's Almanack we are told, that a pound Troy of gold is coined into 44 guineas and an half, and therefore that one guinea should weigh 5 pennyweights $9\frac{39}{89}$ grains.

Also, that a pound Troy of silver is worth $3\frac{1}{10}$ pounds sterling, and therefore that a shilling should weigh 3 pennyweights $20\frac{28}{31}$ grains *.

Are these conclusions true?

A guinea weighs 5 pennyweights $9\frac{1}{2}$ grains, is it over or under the lawful weight, and how much?

In the public stocks 3 per cents. *consol.* are done at $56\frac{5}{8}$; what is the interest of money? Ans. $5\frac{45}{151}$, or a little less than $5\frac{45}{150}$, or $5\frac{3}{10}$, or 5 pounds and 6 shillings.

* A pound Troy contains 12 ounces; each ounce contains 20 pennyweights; each pennyweight 24 grains.

ALGEBRA.

ALGEBRA.

1. ALGEBRA is not improperly considered as a kind of short-hand writing, which may at any time be translated into words at length. Such a short hand is of great use in reducing any reasoning into writing. The argument is more readily comprehended; and the relation between distant ideas much easier discerned, when briefly expressed by single characters, than when encumbered by words at length. We may therefore define Algebra to be the art of reasoning upon Quantity by Symbols, or short-hand characters.

2. The characters which first occur, are the marks for the four operations in Arithmetic. The mark for addition is $+$, and is called *plus* or *more*; thus $5+2$ signifies 5 made more by 2. The mark of subtraction is $-$, and is called *minus* or *less*; thus $5-2$ signifies 5 made less by 2. The mark for multiplication is $\times$, and is called *into*; thus 5×2 is read 5 into 2, meaning 5 multiplied by 2. The phrase *into* is borrowed from the Latin: for *Ducere numerum in numerum* is to multiply one number by another. Division is signified by setting the dividend above, and the divisor below a straight line, like the numerator and denominator of a vulgar fraction, and the phrase for it is, *by:* thus $\frac{5}{2}$ is read 5 by 2, and signifies 5 divided by 2. Lastly, $=$ is the mark of equality, and stands for, *is as much as*, or *makes*. Thus $20+10=30$ when

tranſlated into words at length is, 20 made more by 10 is equal to, or makes, 30. Again, $20-10=10$, when tranſlated, is, 20 made leſs by 10 is equal to 10. Thus alſo, $20\times 10=200$, when tranſlated, is, 20 into 10 (that is, multiplied by 10) is equal to 200. Laſtly, $\frac{20}{10}=2$, when tranſlated, is, 20 by 10 (that is, divided by 10) is equal to 2.

3. In all theſe caſes the ſign governs the number or quantity that follows it. Thus, in the algebraic characters $5-2$, it is the number 2 that is to be ſubtracted.

4. Beſides theſe characters for arithmetical operations, the quantities themſelves, which are the object of mathematical enquiry in any algebraic problem, are marked down, named, or ſignified by the letters of the alphabet. It is no unuſual thing in law, in like manner to diſtinguiſh the parties litigant, by letters of the alphabet, as well as by the names of particular perſons. When a caſe is thus ſtated generally, the letters of the alphabet, not ſtanding for individuals, but for all that come under a certain character and deſcription in law (as plaintiff and defendant, leſſor and leſſee,) the determination of that caſe will be a general one, will become a rule, for all others who come under the ſame deſcription of law. Now this way of repreſenting the ſubjects of enquiry by letters will be found equally uſeful in ſcience. In an arithmetical problem then, theſe letters repreſent certain numbers; as, principal and intereſt, &c. In geometrical problems they repreſent certain lines; as, length and breadth; or ſurfaces, or ſolids. In ſtatical problems they repreſent certain weights: in mechanics, certain forces, &c. Theſe letters then are repreſentatives of quantity in general, or in the abſtract, without reſtraining them to ſignify either any particular degree of quantity, or even any particular ſpecies of quantity. Thus, if 10 pounds be the principal then, at five per cent. I can find, from the rule of three, that 10 ſhillings is the intereſt: but if

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I put a for the principal, be it what it will, and b for the intereſt; if I am told that $a=20 \times b$, I get a general rule for the intereſt in all caſes at five per cent. In like manner a may ſtand for the length, and b for the breadth, of certain figures; and thus general rules may be eſtabliſhed ſhowing, in every one of thoſe figures, the relation of the length to the breadth. Algebraic rules, in which the ſymbols ſtand for ſuch very general ideas, will of courſe be applicable to every ſubject into which mathematical reaſoning can be introduced.

5. All quantities have either the mark of addition or ſubtraction before them, and are called either *affirmative* or *negative*.

This diſtinction of quantities into two ſorts ariſes from the nature of the ſubject to which algebraic computation is applied. For inſtance; if a queſtion be inſtituted concerning how much a perſon is worth in the world, all that he poſſeſſes, or is owing to him, increaſes his worth; and, in eſtimating that worth, is to be added to his ſtock: but all that he owes to others decreaſes his worth, and is to be ſubtracted from his ſtock. Two ſorts of quantities, in this caſe, ariſe from the nature of the ſubject; credits and debts. The former addititious or affirmative, before which you prefix the ſign of addition $+$; the latter is ſubductitious or negative, before which you prefix the ſign of ſubtraction $-$. In computing the worth of ſuch a perſon, if the addititious quantities prevail, his ſtock will be marked $+$, and he will be worth ſomething. If the addititious and ſubductitious quantities exactly balance, he will be worth nothing. If the ſubductitious quantities prevail, his ſtock muſt be marked $-$; or, as we ſometimes ſpeak of ſuch a perſon's worth, he is worſe than nothing. Here it ought to be remarked, that debts, or ſubductitious ſums of money are as much real ſums of money, real numbers, as credits or addititious ſums; the mark $-$ being a mark of the *quality* of ſuch numbers, and not of their

quantity, denoting that their quality is ſuch, that in reaſoning upon the worth of a perſon they are to be ſubtracted.

6. To give another example. Mechanics is the doctrine of the motion of ſolid bodies, when acted upon by various forces. Here the nature of the ſubjects points out two circumſtances that muſt be attended to, the *direction* of the motion excited in the ſolid body by the joint action of the ſeveral forces, and the degree, ſwiftneſs, or magnitude of the motion. The firſt circumſtance reſpects the *quality*, the ſecond the *quantity* of the motion. Suppoſe, for the ſake of ſimplicity, that all the forces are impreſſed in one and the ſame line, but ſome of them backwards and ſome forwards; that is, ſome in one direction arbitrarily aſſumed, which is to be called *forwards*; others in the oppoſite direction, and of courſe to be called *backwards*. In this caſe all the forces urging the body forwards are to be called affirmative, and marked +. All the forces urging the body in the oppoſite direction, or backwards, are to be called negative, and marked −. If in the concluſion of the problem the motion of the body is found affirmative, and marked +, it is a ſign that the affirmative forces prevail, and that though ſome forces ſingly conſidered drive it forwards, and others ſingly conſidered drive it backwards, yet, on the whole, it will move forwards: but if in the concluſion of the problem the motion ſought turns out negative, and marked −, it is a ſign that, on the whole, the body will move backwards. Yet this motion backwards is as much a real motion as any motion forwards; the marks + or − being marks of the quality, not of the quantity, of motion. Negative motion is as much a real motion as affirmative motion, but in an oppoſite direction. If the affirmative and negative quantities ſhould, on the whole, balance, the degree of motion will be found to be o; that is, the body will have no motion at all, or ſtand ſtill. But to talk of any motion, as being

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lefs than nothing, or lefs than no motion, is to fay, that a body may be more ftill than ftock-ftill, which is downright nonfenfe.

7. One doubt may arife on this fubject. It is ufual to exemplify Algebraic rules by numbers. It may be afked then, if we meet with fuch an expreffion as -5 ftanding fingly by itfelf, what does this mean? We have been told, that $-$ is the mark of fubtraction; that when it is prefixed to any quantity it denotes that quantity is to be fubtracted: Now, in this cafe, from what number is 5 to be fubtracted? We anfwer, from the other numbers that arife out of the conditions of the problem; -5 has no meaning unlefs it be joined with fome other number. If no queftion be inftituted; if the number 5 is to be confidered abftractedly, having no reference at all to other numbers, then the marking it either with $+$ or $-$ is unwarranted, and leads to abfurdities.

8. There are mathematical problems which refpect the magnitude only of the quantities concerned; problems in which their quality is no part of the fubject under confideration. Such are all queftions about proportion. Proportion has reference to magnitude only. The idea of proportion refults from contemplating and comparing magnitudes only; their quality hath nothing to do with it: and if the fymbols $+$ and $-$, and the diftinction of quantities into affirmative and negative, are arbitrarily retained, when fuch a diftinction is not warranted by the fubject itfelf, it will lead us into endlefs abfurdities *.

9. We

* So it is reprefented as a wonderful paradox, that -1 fhould be to $+1$, as $+1$ to -1; that is (fuppofing -1 to be a *quantity lefs than nothing*, and of courfe lefs than $+1$) that a lefs quantity fhould have the fame proportion to a greater quantity, as that fame greater quantity has to the lefs. And this is demonftrated, fay they, by the rules of Algebra; for according to thefe rules the product of the multiplication of the two extreme terms in this proportion is equal to the product of the two mean terms. But all

9. We may now return to ſome further obſervations relating to the ſigns or characters uſed in Algebra.

A compound quantity is that which conſiſts of ſeveral quantities connected by the ſigns + or −, as $a-b+c$, and in this caſe the value of the compound quantity is not altered by changing the order in which the ſeveral members are ranged, as $-b+a+c$ or $+c-b+a$, as will appear if we ſubſtitute numbers for the ſeveral letters a, b, and c; but it is uſual to range them in alphabetical order. It is uſual alſo to make one of the affirmative quantities the leading quantity; and in this caſe the ſign + is not expreſſed, or written before the leading quantity, but underſtood, as in the example before given.

10. In eſtimating the numeral value of ſuch compound quantities, it is beſt to reckon up the value of all the affirmative quantities, and then reckon up the value of all the negative quantities. The leſs of theſe two ſums is to be ſubtracted from the greater; the remainder will be the value of the whole expreſſion, and is of the ſame quality, affirmative or negative, with the greater of thoſe two ſums.

11. With regard to the ſign of multiplication, it may be obſerved, that it is not always expreſſed: for if a number and a letter be joined together, or if two letters be joined together like the letters of a word, the ſign of multiplication is then underſtood to be interpoſed between them; that is, the product of the multiplication of the numbers which thoſe letters repreſent is ſignified as well by the letters being thus joined, as by the ſign × being interpoſed between them. Thus, if $a=20$, and $b=10$, then either $a\times b$ or $ab=200$, ſo alſo $5\times a$ or $5a=100$: and in reading ſuch

all this is nonſenſe. In reſpect of magnitude + and −1 are equal; and it is in reſpect of their magnitude only that they can have *proportion* to each other. Proportion has no concern with their quality. They have no quality in a queſtion about their proportion only.

expreſ-

expreſſions, when the ſign $\times$ is actually written the word *into* is pronounced; but that word is omitted when the ſign is not written: ſo $5 \times a$ is read, five into a, but $5a$ is read *five a*; meaning five times a. Numbers thus joined with letters, are called *numeral co-efficients*, or ſimply co-efficients; where no numeral co-efficient is expreſſed 1, or unity, is underſtood; thus a, without a co-efficient, is the ſame as $1a$ or once a.

12. When a quantity conſiſts of the product of the continual multiplication of two, three, or more quantities, the whole product is called a *factum*, and the ſeveral multipliers are called *factors*, and relatively to one another *fellow-factors* or *co-efficients*. So in the quantity abc, the ſeveral quantities a and b and c are called factors, the whole product abc the factum. So again, a and bc are two fellow-factors, or a is the co-efficient of bc in the factum abc, or ſimply in abc. So again, becauſe $4 \times 5 \times 6 = 120$, the numbers 4 and 5 and 6 are called factors, and 120 the factum. And here it muſt be obſerved, that in all ſuch products, conſiſting of three or more factors, the value of the factum is not altered by the order in which the factors are ranged. Thus, abc, bac, or cba, &c. are all of the ſame value. So again, $4 \times 5 \times 6$, or $5 \times 4 \times 6$, or $6 \times 5 \times 4$, &c. are all of the ſame value, and equal to 120; but it is uſual to range literal factors in an alphabetical order.

13. There is another character called a *vinculum*, which is a ſtraight line drawn over two or more quantities thus, $\overline{a+b}$, or $\overline{a+b-c}$; and it denotes that all the quantities under the vinculum are to be conſidered as *one*, in reſpect of any algebraic ſign prefixed to the whole. We meet with this character very commonly in caſes of multiplication. Thus $a \times \overline{a+b}$ means that the number a is to be multiplied into the whole number ſignified by $\overline{a+b}$. If we write this without the vinculum, thus $a \times a+b$, it ſignifies that

a is to be multiplied by a, and b is to be added to that product. So again, $\overline{a+b} \times \overline{a-b}$, signifies that the intire number $\overline{a+b}$, is to be multiplied by the intire number $\overline{a-b}$. Should we write the same characters without the vinculum thus, $a+b \times a-b$, it would mean that a is to be added to the product of the multiplication of b into a, and that the whole sum is to be made less by b, or that from the whole sum you are to subtract b.

14. In reading such algebraic expressions, when the vinculum is over two quantities, it is signified by pronouncing the worth *both*. So $\overline{a+b} \times \overline{a-b}$ is read $\overline{a+b}$ both, into $\overline{a-b}$ both. When the vinculum is over three or more quantities, it is signified by pronouncing the word *all*. Thus $\overline{a+b-c} \times \overline{d-e+f}$ is read $\overline{a+b-c}$ all, into $\overline{d-e+f}$ all.

15. A sign set before a fractional quantity belongs to, and governs, the whole of that quantity; it belongs not to the numerator only, or to the denominator only; nor does it belong to a particular member of the numerator, nor to a particular member of the denominator, but to the whole fraction. Thus $\frac{a+b}{12} \times \frac{a-b}{20}$, signifies that the whole fraction $\frac{a+b}{12}$ is to be multiplied into the whole fraction $\frac{a-b}{20}$. So again, $\frac{a+b}{12} + \frac{a-b}{20}$ signifies that the whole fraction $\frac{a+b}{12}$ and the whole fraction $\frac{a-b}{20}$ are to be added together. In reading such fractions whose numerators are compound quantities it is usual to pronounce the words *both* or *all*, just as if they had a vinculum over them. Thus, the last expression is to be read $\overline{a+b}$ both by 12, plus $\overline{a-b}$ both by 20.

Other characters or rules relating to algebraic notation will be explained when they occur.

16. Ex-

16. Examples of algebraic expreſſions; the numeral values of which are to be computed, in order to learn perfectly the power or force of the algebraic characters.

Let $a=10$
$b=7$
$c=2$
$d=1$
$e=0$.

Then 1 $aab+c-d=$

2 $5ab-10bb+e=$

3 $\frac{aab}{c}\times d=$

4 $\frac{a+b}{c}\times\frac{b}{d}=$

5 $\frac{a+b}{c}-\frac{a-b}{d}=$

6 $\frac{aab}{c}+e=$

7 $\frac{aab}{c}\times e=$

8 $\overline{b-c}\times\overline{d-e}=$

9 $\overline{a+b}-\overline{c-d}=$

10 $\overline{a+b}-c-d=$

11 $aac+ddd=$

12 $acd-d=$

13 $aae+bbe+d=$

14 $\frac{b-e}{d-e}\times\frac{a+b}{c-d}=$.

Of what is called ADDITION of Algebraic Quantities.

We ſhall firſt lay down the rule and illuſtrate it by examples.

17. Here it muſt be premiſed, that thoſe quantities are ſaid to be of the ſame denomination, which are exactly

exactly alike in their literal or unknown part, and differ only in their known part, that is, in their ſigns and numeral co-efficients. Thus, $10aa$ and $-8aa$ are of the ſame denomination: but $10aa$ and $-8a$ are not of the ſame denomination. So again, $12aabb$ and $-aabb$ are of the ſame denomination; but $12aabb$ and $-aab$, are not of the ſame denomination. Once more, $5abc$ and $-5abc$ are of the ſame denomination; but $5abc$ and $-5ac$ are not of the ſame denomination. Thus alſo, the number 50 and the number 60 may be ſaid to be of the ſame denomination; being both known quantities; but the number 50, and the letter *a* (even though *a* ſtands for a number, and not a line) are not of the ſame denomination, the one being wholly known, the other wholly unknown. This diſtinction underſtood, we have the following *Rule*.

18. Range all the quantities of the ſame denomination underneath one another; ſetting down the others at length, either in the firſt line, or in any other, and ſupplying their places in the lines in which they are not entered, with aſteriſks. This done, the quantities of the ſame denomination are to be *algebraically* added together by their numeral co-efficients. If their co-efficients have like ſigns; add them together, arithmetically; their common ſign is the ſign, their ſum, the co-efficient, to be prefixed to the literal part that belongs to them all. If the ſeveral quantities of the ſame denomination have unlike ſigns; then all the affirmative co-efficients muſt be collected into one ſum, and all the negative co-efficients into another ſum; and the difference of theſe two ſums is the co-efficient of the common literal or unknown part; to which muſt be prefixed the ſign of the greater of thoſe two ſums. Thus, like ſigns require an addition of the numeral co-efficients; unlike ſigns a ſubtraction of thoſe co-efficients.

19. Quantities that are not of the ſame denomination

tion can no otherwiſe be added, than by ſetting them down at length with their proper ſigns.

20. In the practice of this and the following algebraic rules, it is beſt firſt to determine the ſign, then the co-efficient, and laſtly, the letters.

21. In the following examples the quantities are not properly ranged under each other; *that* is left as an exerciſe for the learner.

1. To $b+a$ add $3a-5b$
2. To $-4x+3a$ add $5a-8x$
3. To $6x-5b+a+8$ add $-5a-4x+4b-3$
4. To $a+2b-3c-10$ add $3b-4a+5c+10$, alſo $-4c+2a-3b-7$, alſo $+5b-c$
5. To $3a+b-10$ add $c-d-a$
6. To $3aa+bb-c$ add $2ab-3aa+bc-b$
7. To $aaa+bbc-bb$ add $abb-abc+bb$
8. To $9a-8b+10x-6d-7c+50$ add $2x-3a-5c+4b+6d-10$
9. To $a+b$ add $a-b$, and the ſum is $2a$.

22. In the algebraic language a and b may ſtand for any two numbers whatever; and then $a+b$ ſtands for a made more by b; that is, for the ſum of a and b; again, $a-b$ ſtands for a made leſs by b; that is, for the difference of a and b (where b is ſuppoſed leſs than a). Now, by the rules of algebra, $a+b$ and $a-b$ added together make $2a$; therefore if the ſum and difference of any two numbers be added together; the whole will be $2a$, or twice the greater number. This we ſhall find on trial holds good in any inſtances whatever.

Remarks on Algebraic Addition.

23. The word *addition* here is very improperly uſed; and is vaſtly too ſcanty to expreſs the operation here performed. The buſineſs of this operation is to incorporate into one maſs (or algebraic expreſſion) different algebraic quantities; as far as an actual incorporation or union is poſſible; and to retain the

 algebraic

algebraic marks for doing it, where it is impoſſible. When we have ſeveral quantities, ſome affirmative and ſome negative; and the relation of theſe quantities can in the whole or in part be diſcovered; ſuch incorporation of two or more quantities into one is plainly effected by the foregoing rule. Thus, if ſeveral numbers are ſet down, ſome to be added, others to be ſubtracted; their relation is known, and they may be united into one equivalent number, by adding or ſubtracting as their ſigns import. When the literal part of two algebraic expreſſions are alike, but their numeral co-efficients are different; the relation of two ſuch quantities is in part known, in part unknown. With reſpect to the literal part; although the number or quantity for which theſe letters ſtand is not known, yet this circumſtance is plainly told us, that the unknown part of all theſe expreſſions is the ſame and alike, and their co-efficients being numbers are fully known. Such quantities therefore may in part be incorporated, as far as they are known; that is, by their co-efficients as before taught. But if the literal part of two algebraic expreſſions be different; if no degree of relation between them can be diſcovered; then no actual incorporation can take place, either in part or in whole. A character for expreſſing that incorporation, and the nature of the arithmetical operation by which it is to be performed (when thoſe quantities become known) is all that can take place. So if $-b$ is to be *algebraically* added, that is to ſay incorporated with $+a$, we can only, at preſent, write down the characters $a-b$; ſignifying that a is to be added (to the other quantities in the problem), and b is to be ſubtracted (at their final incorporation) when their values are known.

24. The relation between two abſolute numbers is always evident and wholly known; and ſuch can always be incorporated or united into one equivalent number. So two affirmative or addititious numbers may be incorporated into one greater affirmative number,

ber, equivalent to the former two. So alſo two negative or ſubductitious numbers may be united into one greater negative number. The operation (of uniting) in both theſe caſes, where the ſigns are alike, is performed by the arithmetical addition of theſe numbers.

25. If one number be affirmative and the other negative, theſe may alſo be united into one equivalent number. Theſe numbers in their application denote the magnitude of certain quantities, but having oppoſite qualities; therefore, in their union, the leſs muſt be taken from the greater; and there will remain one number (or magnitude) equivalent to the other two; and of the ſame quality with the larger of thoſe two. Thus then two numbers; one affirmative, and one negative; may be united into one equivalent number. In the algebraic language this is called *adding* an affirmative and negative number together, although the buſineſs is carried on by the arithmetical operation of ſubtraction. So to ſpeak in the algebraic language, the ſeveral numbers, $+10$ and -8, and $+7$ and -6, all four *added* together make $+3$; and the number $+3$ is called the *ſum* of thoſe four numbers, although ſuch ſum is made up, partly by the arithmetical operation of addition, and partly by the arithmetical operation of ſubtraction.

26. It may ſeem a paradox, that what is called addition in algebra, ſhould ſometimes mean addition and ſometimes ſubtraction. But the paradox wholly ariſes from the ſcantineſs of the name given to the algebraic proceſs; from employing an old term in a new and more enlarged ſenſe. Inſtead of addition, call it incorporation, union, ſtriking a balance, or any name to which a more extenſive idea may be annexed than that which is uſually implied by the word addition; and the paradox vaniſhes.

THE RULE FOR ALGEBRAIC SUBTRACTION.

27. Write down in one line thoſe quantities from which the ſubtraction is to be made, and which we will call the firſt propoſed quantities; then underneath write down all the quantities compoſing the ſubtrahend, or ſecond propoſed quantities; ranging the quantities of the ſame denomination under each other, as before directed. Change now, or rather ſuppoſe to be changed, the ſign of every quantity in the ſubtrahend, and proceed by the rules before given for addition. Examples as in par. 21.

1. From $4a-4b$, take $b+a$
2. From $4a-4b$, take $-5b+3a$
3. From $8a-12x$, take $-4x+3a$
4. From $2x-4a-b+5$, take $8-5b+a+6x$
5. From $3a+b+c-d-10$, take $c+a-d$
6. From $3a+b+c-d-10$, take $b-10+3a$
7. From $2ab+bb-c+bc-b$, take $3aa-c+bb$
8. From $aaa+bbc+abb-abc$, take $bb+abb-abc$
9. From $12x+6a-4b+40-12c$, take $4b-3a+2x+6d-7c-10$
10. From $2x-3a+4b-5c+6d-50$, take $9a+x-7c+8b-6d-40$
11. From $6a-4b-12c+12x-7e-5f$, take $2x-3a+4b-5c+6d-7e$
12. From $a+b$, take $a-b$, and there remains $2b$.

28. We have before taken notice, that $a+b$ may repreſent the ſum, and $a-b$ the difference of any two numbers; of which a is the greater and b the leſs. Now here it appears, that if $a-b$ be ſubtracted from $a+b$, the remainder will be $2b$; that is, if the difference of any two numbers be ſubtracted from their ſum, the remainder will be twice the leſs number.

29. Thus, by means of the algebraic character and rules, we have got two general Theorems or Truths, relative to the ſum and difference of any two numbers. We might have found theſe theorems to be true in every particular inſtance we could try, and thence

thence have presumed them to be true in every untried instance; but this is only a *presumption* grounded on what the logicians call an induction of particulars: whereas the algebraic process affords a general demonstration, extending to all cases whatever.

Remarks on Algebraic Subtraction.

30. As we have affixed a more extensive idea to the word addition in algebra, than what is given to that term in common arithmetic; so we must in correspondence thereto, give a more enlarged idea of the word *Subtraction*. The business of subtraction then, is to find such a quantity as being algebraically added to, or united with, the second of two proposed quantities, will produce the first. Now that the rule before laid down is just, may be demonstrated. Because if the quantity found by that rule, called improperly the Remainder, be actually united with the second of the two proposed quantitses, it will always produce the first proposed quantity. *Ex.* From $5a+4b$, let it be proposed to subtract $3a-2b$. The answer according to the rule is $5a+4b-3a+2b$. And that it is the true answer is proved by adding it to the subtrahend. For $5a+4b-3a+2b$ added to $3a-2b$ produces the quantity first proposed, viz. $5a+4b$. But the above rule (as far as relates to the changing of the signs from $+$ into $-$ and $-$ into $+$) may be illustrated, and perhaps established to the satisfaction of the learner on other considerations. For this in fact is no other than what is constantly practised in all book-keeping. If a debt is paid off, it is not the custom to erase the entry of that debt; nor to subtract the debt paid off, from the sum total of debts; but to enter the sum paid, on the creditor side; in the merchant's phrase to give, such a one, *Credit*, for cash received. Whenever the balance is struck, the account will be as fairly made up, by making such an entry into the account, as if an erasure

ſure had been made. So if we are to ſubtract -5 from $+8-3$, we enter it in this form $+5+8-3$. If we are to ſubtract $c-d$ from $a-b$, we add or unite $-c+d$ to $a-b$, and enter it in this form $a-b-c+d$. Whenever the numeral values of a and b and c and d can be found, and thoſe values united into one number according to their ſigns, the balance will be rightly ſtruck. Here we enter c with the mark of ſubtraction, inſtead of an actual or arithmetical ſubtraction of the affirmative quantity c from the affirmative quantity a: and we enter d, with the mark of addition, inſtead of an arithmetical ſubtraction of the value of $-d$, from the arithmetical value of $-b$; for ſuch an arithmetical ſubtraction is impoſſible, while the values of a, b, c, and d, are unknown. When no relation between them can be diſcovered, they cannot be made to incorporate by an arithmetical addition.

31. In the ſame manner, if from $10a-b$ we ſubtract $2a$, we enter it, as if it were $10a-2a-b$, but then we incorporate $10a$ and $-2a$ into one quantity $8a$, ſo the whole is $8a-b$. Here, inſtead of expunging $2a$ out of the creditor ſide of the account, we enter $2a$ on the debtor ſide; and afterwards ſtrike a balance.

32. Every ſubtraction therefore is an entry of the ſeveral quantities to be ſubtracted with contrary ſigns; and all the quantities ſo entered, are to be algebraically added or united with the reſt. When we would ſubtract an affirmative quantity, inſtead of an arithmetical ſubtraction (which is impoſſible in ſymbols) we enter or write the ſame quantity or ſymbol with the contrary ſign *minus*. When we would ſubtract a negative quantity, we enter that quantity with the ſign *plus*; and theſe quantities ſo entered with changed ſigns, are to be algebraically added. This being obſerved will in a great meaſure help us over that almoſt inſuperable difficulty;—the reaſonableneſs of the rule for the change of ſigns in multiplication.

OF

OF ALGEBRAIC MULTIPLICATION.

33. In all algebraic quantities we have before noted three circumſtances, the ſign, the numeral co-efficient, and the literal or unknown part. The following are the rules which obtain in multiplication with reſpect to each of thoſe circumſtances.

34. Firſt, like ſigns give +, unlike ſigns − in the product. Secondly, let the numeral co-efficients of the two factors be multiplied together; their product will be the numeral co-efficient of the factum. Thirdly, the letters of the multiplier and multiplicand being placed together, like the letters of a word, make the literal part of the product. We ſhall poſtpone, for the preſent, the grounds of the rule reſpecting the ſigns. The rule reſpecting the co-efficients, and the literal part is ſelf-evident. As far as the quantities are known, that is, in their numeral co efficients, an arithmetical multiplication takes place. In the literal part, where the quantities repreſented by the letters are unknown, the character for ſuch multiplication is uſed. That character, as we ſaid before, is the ſetting together the letters which repreſent the unknown quantities, like the letters of a word.

35. If the multiplicand be a compound quantity, and the multiplier a ſimple quantity, then every member of that compound quantity muſt be multiplied by the multiplier. For the whole product conſiſts of the product of the multiplication of every part or member of the multiplicand by the multiplier.

36. If the multiplier be a compound quantity as well as the multiplicand, then the whole multiplicand muſt be multiplied by every member of the multiplier. In this caſe, multiply the whole multiplicand by the firſt member of the multiplier, and ſet down the product in one line. Proceed then to multiply the whole multiplicand by the ſecond member of the multiplier, but in ſetting down the ſeveral products

as they now arise, range them under those quantities in the first line, which are of the same denomination. If no such can be found in the first line, then set these products at length either in the first or second line, not setting them under any that are not of the same denomination. These several lines algebraically added together make the whole product.

37. In writing down the multiplicand and the multiplier, there is no necessity for ranging quantities of the same denomination under each other, or observing any particular order; but in setting down the several products as they arise, that rule must be observed, because those several products are to be added together.

40. Examples of algebraic multiplication:

1. Mult. $4ab$ by $5a$
2. Mult. $-6a$ by $+7b$
3. Mult. $3a$ by $-a$
4. Mult. $6ab$ by $-4b$
5. Mult. $-7a$ by $-b$
6. Mult. $5aa$ by -10
7. Mult. $3a-4b+5c$ by $+2a$
8. Mult. $9a-10b-120$ by $-4a$
9. Mult. $6a-7b-8c$ by $2a-3b+4c$
10. Mult. $aa+ab+bb$ by $a-b$.

Definition of the Terms POWERS *and* INDEXES.

41. The products that arise from the continual multiplication of one number by itself are called *powers* of that number. Thus 3, 9, 27, 81, 243, are powers of the number 3. Also, a, aa, aaa. &c. are powers of the number represented by the letter a. See Tab. II. and Tab. IV.

Table I.	Indexes	1	2	3	4	5	6	7	8
	Powers	2	4	8	16	32	64	128	256

Tab. II.	Indexes	1	2	3	4	5	6	7
	Powers	3	9	27	81	243	729	2187

Tab. III.	Indexes	1	2	3	4	5
	Powers	10	100	1000	10000	100000

Tab.

Tab. IV.						
	Indexes		2	3	4	5
	Powers	a	a	a	a	a
	Powers	a,	aa,	aaa,	$aaaa$,	$aaaaa$.

42. If the series of the powers which thus arise from the continual multiplication, of one and the same number or quantity by itself, be marked with the natural numbers, 1, 2, 3, 4, 5, &c. in order; those numbers are called indexes of the powers to which they belong, and they are read thus (Tab. II.), 3 in the power of 1 equals 3; 3 in the power of 2 equals 9, and 3 in the power of 3 equals 27, and 3 in the power of 4 equals 81. And here we must carefully distinguish between two expressions, somewhat alike in sound, but very different in sense: 4 times 3 means 3 taken 4 times, or fourfold, and is equal to 12. But 3 in the power of 4 (or, as it is sometimes called, the fourth power of 3) means 3 multiplied by itself 4 times, or 4 multiplications of 3, and is equal to 81.

43. Powers are also thus denominated from their indexes. The number itself is called the First power. The number multiplied by itself is called the Second power of that number. This product again multiplied by the original number is called the Third power, and so on. The first power is also called the Root, the second power is called the Square, the third power is called the Cube. These are names borrowed from Geometry, but are of frequent use.

These Indexes have remarkable properties, which it is necessary for us here to enumerate.

44. First: the addition of indexes answers to the multiplication of the powers to which they belong. That is, if the index of any power of a number be added to the index of another power of the SAME number, their sum will be the index of a power, which is the product of the multiplication of the two former powers. Thus, in Tab. II, which is a scale of the powers of the number 3, the number 2 is the index of the number 9 among the powers of 3, and 3 is the index of the number 27 among the powers of

the ſame number 3. Now add theſe two indexes together, and their ſum is 5, which is the index of 243 among the powers of 3. Therefore we may be ſure, that 9 times 27 make 243. So again, 3 is the index of 27, and 4 is the index of 81, in the ſame ſcale of the powers of 3. But 3 and 4 added together make 7. Look for 7 then in the ſcale of indexes, and under it we ſhall find 2187 in the ſcale of powers. Therefore 2187 is the product of the multiplication of 27 by 81.

45. We have ſeen then, that the addition of indexes anſwers to the multiplication of powers. It follows of courſe, that ſubtraction of indexes muſt anſwer to the diviſion of powers. That is, if the index of the diviſor be ſubtracted from the index of the dividend, the remainder will be the index of the quotient. Thus in the ſcale of powers of the number 3, if we would divide 2187 by 243, ſubtract 5, the index of the diviſor, from 7, the index of the dividend, and the remainder 2 will be the index of the quotient, which is therefore 9, as appears by inſpecting the table of the powers of 3.

46. As the ſquaring any number is the multiplying that number by itſelf, ſo the ſquare of any number (in the ſcale of powers) will be found by adding the index of that number to itſelf, that is, by doubling the index of that number. Thus, the index of 27 in the ſcale aforeſaid is 3, and 3 doubled is 6, which is the index of 729, the ſquare of 27. In like manner cubing is performed by tripling the index. Thus, 2 is the index of 9, and 2 tripled is 6, the index of 729, which is the cube of 9.

47. But what is more material than all this; it follows from the laſt property (viz. that multiplication of indexes anſwers to the raiſing of powers), it follows, I ſay, that diviſion of indexes muſt anſwer to the extraction of roots. To extract the ſecond, third, or fourth, root of any number, is to find that number, which being multiplied by itſelf two times, 3 times, or 4 times, will produce the original or propoſed

posed number. Thus 3 is the second root of 9, and it is the third root of 27, or as we most commonly say, 3 is the square root of 9, and the cube root of 27. If then we want to extract the cube root of 729, we divide its index 6, by the number 3, the quotient 2, is the index of the cube root of 729, therefore this root is 9. For as 6 divided by 3, gives a number, which added to itself 3 times, or multiplied by 3, produces 6; so if the index of 729 be divided by 3, it will give the index of a power (in the scale of powers, Tab. II.) which multiplied by itself 3 times, or raised to the third power, will produce 729.

When letters represent numbers, the index is put after the letter, and in a less character and above it. This distinguishes the index from the co-efficient. Thus $6a^2$ is six times a in the power of 2.

48. Examples of multiplication where indexes occur.

1. Mult. a^2b by $3a^3b^2$
2. Mult. ab by $-ab$
3. Mult. $4ab^2$ by $-5ab^3c$
4. Mult. $-abc$ by $-4a^2b^3c^4$
5. Mult. $a^2b^2+3abc-7a$ by $-5a^3c$
6. Mult. $a^3+a^2b+ab^2+b^3$ by $a-b$
7. Mult. $6a^2-7ab+c^2$ by $2a^2-3ab-4c^2$
8. What are the 2d, 3d, and 4th, powers of $a+b$?
9. Raise $a-b$ to the 4th power.

49. It has been observed, that $a+b$ may represent the sum, and $a-b$ the difference of any two numbers; whereof a is the greater and b the less. Multiply $a+b$ by $a-b$, and the product is $aa-bb$, but aa is the square of the greater, and bb is the square of the less, and $aa-bb$ is their difference. Hence we get this theorem, that if the sum of any two numbers be multiplied by their difference, the product will be the difference of the squares of those two numbers.

50. It is now time that we should explain the grounds of the rule before laid down respecting the signs in multiplication, namely, that like signs give +, unlike signs − in the product. This is generally

represented as a great mystery, especially when both signs are negative; and indeed the explanations sometimes given make it mysterious enough. Such are those which require you to admit as a *postulatum*, that a negative quantity is a quantity less than nothing; less than no quantity. The best explanations commonly given amount to no more than this, that, unless you admit the rule, you will make an error, but do not lay open the foundation of the rule. Let us try then whether we can explain, or rather demonstrate this rule with better success.

51. Now though the letters in algebra, *a*, *b*, *c*, &c. may for the most part represent any *kind* or sort of quantity (as quantity of units, quantity of extension, quantity of duration, &c.), yet in the business of multiplication, where *ab* stands for a product, we are restrained in the use of these symbols, and must interpret one of them, namely the multiplier, to mean or stand for a *number*. It can stand for nothing else. For multiplication is, rightly defined, *a compendious addition*; the multiplier signifying how many times the multiplicand is to be added to itself. The answer to the question, *how many times?* is plainly a number and can be nothing else. Thus, a yard in length may be taken three times, an hour in duration may be taken three times; but a yard cannot be taken a yard-times, or a foot-times; nor can an hour be taken a day-times, or an hour-times. So also twenty shillings may be taken three times or four times; but cannot be taken twenty-shillings-times. The school question, "multiply twenty shillings by twenty shillings," admits of no answer, because it has no meaning.

52. This premised*, when the multiplier is an affirmative

* If it be objected to what has been said, that we meet with the expression *aa*, where *a* stands for a line; I answer, that this expression *aa*, may also stand for a line,—for a line which is a Third

firmative number, the meaning of ſuch a multiplication is, that we are to add the multiplicand ſo many times together, or to the other quantities in the problem, as there are units in the multiplier. In this caſe, if the multiplicand be a compound quantity, conſiſting of affirmative and negative members, the addition of ſuch algebraic multiplicand muſt be performed by the rule before laid down for the algebraic addition of ſuch compound quantities. The addition of the ſame affirmative quantity ſeveral times, is equivalent to the addition of one greater affirmative quantity. The addition of the ſame negative quantity ſeveral times, is equivalent to the addition of one greater negative quantity; that is, when the multiplier is affirmative, the rule is, that + into + gives +, but + into − gives −.

53. When the multiplier is a negative number, the meaning of ſuch a multiplication is, that we are to ſubtract the multiplicand ſo many times from the other quantities in the problem. The multiplicand is to be accounted ſo many times ſubductitious as there are units in the multiplier. In this caſe, if the multiplicand be a compound quantity, conſiſting of affirmative and negative members, the ſubtraction of ſuch an algebraic multiplicand muſt be performed by the rule before laid down for the ſubtraction of compound algebraic quantities, that is, it muſt be performed by the algebraic addition of thoſe quantities with their ſigns changed. This algebraic addition with ſigns changed is to be repeated as many times as there are units in the multiplier. Thus then, when the multiplier is a negative number, and the multiplicand affirmative, the product muſt be entered down with the ſign −; but if the multiplicand be alſo negative, the product muſt be entered down with the

Third proportional, to a line aſſumed at pleaſure and called unity, and to the line a.

This and other caſes of the like ſort will be better underſtood hereafter.

ſign +. And theſe ſeveral quantities ſo entered muſt be algebraically added together according to the rule before given for ſubtraction in algebra. Thus we ſee that − into + gives −; but − into − gives +. Or, like ſigns give +, unlike −; ſee par. 34.

THE RULE FOR DIVISION.

54. Where the diviſor is a ſimple quantity and the dividend either ſimple or compound; but of ſuch a ſort that the letters of the diviſor are found in every member of the dividend: in this caſe, divide every member of the dividend ſeparately, by the diviſor, according to the following rule.

55. Firſt; Like ſigns give +, unlike ſigns −, in the quotient. Secondly; Divide the numeral co-efficient of the dividend by the numeral co-efficient of the diviſor, and it gives the numeral co-efficient of the quotient. Thirdly; The literal part of the quotient will conſiſt of thoſe letters in the dividend, which are over and above what are found in the diviſor. When powers of the ſame quantity occur, diviſion is performed by the ſubtraction of their indexes, as was before ſhewn, par. 45.

56. The truth of all theſe rules may be ſhown from this principle; that the diviſor multiplied by the quotient produces the dividend.

57. Examples of ſimple diviſion.

1. $3ab)12aab - 15abb + 18aabb($
2. $-10b)20ab - 30abb - 40b($
3. $-20)20ab - 40aabb + 60($
4. $2a^4) - 4a^7 + a^6b - 8a^{10}b^3($
5. $-40ab) - 80ab + 120abb($
6. $-5a^3b^4) - 15a^3b^5 + 20a^4b^4 - 25a^3b^4($

58. In all other caſes, that is, where the diviſor is a compound quantity, or if the diviſor be a ſimple quantity, but its letters not found in *every* member of the dividend, then diviſion is performed by throwing the quotient into the form of a vulgar fraction, thus;

Write

Write down the dividend for the numerator of the fraction, and the divisor for its denominator. This indeed is not an actual division; it is only creating an algebraic character or representation of the quotient. But as the quotient is thus thrown into the form of a fraction, it will be subject to all the rules of vulgar fractions. Thus, two such quotients may be added together, subtracted from each other, &c. and although the original quantities may be unknown, yet the relation which the sum or difference of two such quotients have to the original quantity may be thus discovered.

59. To give an example of this. Divide a by $a+b$ and the quotient expressed fractional-wise will be $\frac{a}{a+b}$. Again, divide b by $a-b$, and the quotient is $\frac{b}{a-b}$. Add these two quotients together, and their sum will be $\frac{aa+bb}{aa-bb}$. Thus, without knowing the numbers which a and b represent, we discover, that if the greater of them be divided by their sum, and the less by their difference, and those two quotients added together, their sum will be equal to the sum of the squares of the two numbers divided by the difference of their squares. Example.

Let the greater number	$a=12$
the lesser number	$b=8$
Their sum	$a+b=20$
The difference	$a-b=4$

The greater divided by their sum, or $\frac{a}{a+b}=\frac{12}{20}=\frac{3}{5}$

The lesser divided by their difference, or $\frac{b}{a-b}=\frac{8}{4}=2$

The sum of these two quotients	$=2\frac{3}{5}$
Now the square of the greater, or	$aa=144$
the square of the less, or	$bb=64$
the sum of their squares	$aa+bb=208$
the difference of their squares	$aa-bb=80$

And 208, divided by 80, is $2\frac{3}{5}$ as before.

OF

OF EQUATIONS.

60. An equation is a propoſition wherein two algebraic expreſſions are declared to be equal to each other. Such algebraic expreſſions uſually conſiſt partly of known and partly of unknown quantities. Thus $2x+9=3x+4$ is an equation where x ſtands for ſome number unknown, conſequently $2x$ and $3x$ will be unknown quantities; the numbers 4 and 9 are known quantities. In this caſe $2x+9$ is ſaid to *poſſeſs* one ſide of the equation, and $3x+4$ is ſaid to *poſſeſs* the other ſide. This equation tranſlated out of algebra into common language would run thus: "Twice a certain unknown number with 9 over, is as much as three times the ſame unknown number with 4 over." And here, by the bye, we may obſerve, that all algebraic expreſſions or equations (which are not nonſenſical, but have a real meaning) can be tranſlated into common language. If they have a meaning, words undoubtedly may be found to expreſs that meaning, ſince algebra is only (as we have ſaid) a ſhort-hand character. We can diſcover the relation between different quantities more clearly, and deduce conſequences from that relation more eaſily, when thus expreſſed in a brief way by ſingle characters, than when written down by words at length. We can alſo apply the rules of algebra reſpecting equations, &c. which are rules to aſſiſt us in reaſoning on ſuch quantities, ſo that we may neither make falſe concluſions, nor uſeleſs ones; ſuch as have no tendency to diſcover the particular truth ſought in any problem: all which will abundantly appear from what follows.

61. To reſolve an equation is to find the relation between the known and unknown parts, or to find the value of the unknown quantity. There are four *proceſſes*, or rules, which are to be applied in the ſame

way,

way, and in the same *order*, for the resolution of all equations; and which will (as it were mechanically) determine the value of the unknown quantity. These processes are grounded upon four axioms or self-evident truths. We shall lay down the four axioms, also the four processes; then explain and demonstrate the truths of those processes in their order.

62. Axiom 1. If equal quantities be added to equal quantities, their sums will be equal.

Axiom 2. If equal quantities be subtracted from equal quantities, their remainders will be equal.

Axiom 3. If equal quantities be multiplied by equal quantities, their products will be equal.

Axiom 4. If equal quantities be divided by equal quantities, their quotients will be equal.

63. Process 1. Clear the equation of fractions.

Rule. Multiply the whole equation by the denominator of any one of those fractions, and that fraction will be taken away. Repeat this operation for every other fraction till the whole equation consists of integral quantities only.

64. Process 2. Bring all the unknown quantities to one side of the equation, namely, to that side which makes them affirmative when incorporated into one unknown quantity.

Rule. This is done by *Transposition*, to be explained presently. The unknown quantities being brought to one side of the equation must be incorporated into one; whence will arise another equation, of the same value, in simpler terms.

65. Process 3. Bring all the known quantities to the other side of the equation.

Rule. This also is done by Transposition. The known quantities being on one side of the equation must, as far as is possible, be incorporated together; whence will arise a simpler equation of the same value.

66. Process 4. When the unknown quantity is joined with a known co-efficient, divide the whole equa-

equation by that co-efficient, and you will have the *single* value of the unknown quantity in known terms.

To explain process 1 we must premise the following Lemma *.

67. If any fraction is to be multiplied by its denominator, this may be done by taking away the denominator, and considering the numerator as integral. Thus, if the fraction $\frac{2}{3}$ is to be multiplied by 3 the product will be 2 integral. Or if $\frac{12}{5}$ is to be multiplied by 5 the product will be 12 integral. And for the same reason, if the fraction $\frac{2x}{3}$ is to be multiplied by 3 the product will be $2x$. So also if $\frac{a+b}{a-b}$ is to be multiplied by $a-b$ the product will be $a+b$.

From par. 39, in the doctrine of fractions, it appears, that if any fraction is to be multiplied by its denominator, this may be done (as we have said) by taking away the denominator, and considering the numerator as integral. An example will make this plain. Let the fraction be $\frac{12}{5}$. Here, following the rule for multiplication in par. 39, the product will be $\frac{12}{1}$; but the fraction $\frac{12}{1}$ is the same as the number 12 integral, by par. 6. Therefore the product of the multiplication of the fraction $\frac{12}{5}$, by its denominator 5, is the numerator 12 considered as integral. It is evident the same will be true of any other fraction. For the denominator divided by itself is always unity; and a fraction whose denominator is unity, is the same with the numerator integral.

Otherwise. Every fraction may be considered as the quotient of the numerator divided by the denominator;

* A Lemma is a proposition previously necessary to the demonstration of another proposition.

minator; see par. 18. But it is a known rule in division, that if the quotient be multiplied by the divisor it produces the dividend; therefore, if the quotient of the numerator divided by the denominator, or the intire fraction, be multiplied by the divisor or denominator, the product of that multiplication will be the dividend, or numerator considered as integral.

68. This premised, we proceed to demonstrate the first process. That this process is just appears from the third axiom; for both sides of the equation are to be multiplied by the denominator of the fraction; if, therefore, they were equal before, they will be equal now. That this process answers the purpose for which it is used appears from the lemma.

69. *A compendium in some cases.* When several fractions occur in an equation, and the denominator of some one of them will *measure* (that is, divide without a remainder) the denominator of any of the others, begin to multiply the whole equation by the denominator of that fraction, and let the fractions, whose denominators can be so measured, be multiplied by dividing their denominator by the number which thus measures them; as was shewn in the doctrine of fractions, par. 39.

OF TRANSPOSITION.

70. Transposition is the removing of any quantity from one side of the equation to the other, *changing its sign*; which may always be done without destroying the equality; that is, the new equation which results from such transposition will be a true equation. Thus, let $a-b=c-d$. Remove now $-b$ to the other side of the equation, changing its sign, and we have $a=c-d+b$, which I prove to be a true equation thus:

By

By the hypothesis $a-b=c-d$
It is self-evident, that - $+b= \quad +b$
Add the latter equat. to the former, and $a* \quad =c-d+b$
which is a true equation by the first axiom.

Again; By the hypothesis $a-b=c-d$, transpose a, and we have $-b=c-d-a$, which I thus prove:
By the hypothesis - $a-b=c-d$
It is self-evident, that - $a \quad = \quad a$
Subtract the latter equation from the former, and we have $\Big\} *-b=c-d-a$
which is a true equation by the second axiom. Thus transposing a negative quantity is adding that quantity to both sides of the equation. Transposing an affirmative quantity is subtracting that quantity from both sides of the equation. The former operation makes the value of the whole equation greater, the latter operation makes the value of the whole equation less than before by the quantity so transposed.

71. In the second process it is said, that all the unknown quantities must be brought to that side of the equation which makes them affirmative when incorporated into one. Therefore, when there is an unknown quantity on each side of the equation a question will arise, which of the two are we to transpose? The following is the answer. If both the unknown quantities are affirmative transpose the least. If both are negative transpose the greatest. If one be affirmative and the other negative transpose the negative quantity. What we have now said explains and proves the truth of process second and third.

72. The truth of the fourth process is evident from the fourth axiom. That it will answer its end, and give the *single* value of the unknown quantity, is plain, because if any quantity be divided by itself the quotient will be unity or one.

Examples

Examples of the Resolution of simple Equations.

73. Example 1. Let $\frac{2x}{3}+4=\frac{7x}{12}+9$.

2. By process 1. $2x+12=\frac{21x}{12}+27$.

3. By process 1. $24x+144=21x+324$.
4. By process 2. $24x+21x+144=324$.
That is (by the same), $3x+144=324$.
5. By process 3. $3x=324-144=180$.
6. By process 4. $x=60$.

Proof. $2x=120$, $\frac{2x}{3}=40$, $\frac{2x}{3}+4=44$. Again, $7x=420$, $\frac{7x}{12}=35$, $\frac{7x}{12}+9=44$, as before.

74. If in the first step we had made use of the compendium before laid down, the work would have stood thus: Original equation $\frac{2x}{3}+4=\frac{7x}{12}+9$

$$2x+12=\frac{7x}{4}+27$$
$$8x+48=7x+108$$
$$8x-7x+48=108$$
that is $$x+48=108$$
$$x=108-48=60,$$ as before.

75. Let us now enquire into the arithmetical value of every step, to see whether the equality is kept up.

Now if x be 60, then the value of each side of the original equation is 44, as was before shown. The second step is got by multiplying the first step by 3. Now $2x=120$, and $2x+12=132$. Again, $21x=1260$, and $\frac{21x}{12}=105$, and $\frac{21x}{12}+27=132$, as before.

fore. Here then both ſides are equal to 132, which is three times 44.

The third ſtep is derived from the ſecond, by multiplying the whole equation by 12. Now $24x = 1440$, and $24x + 144 = 1584$. Again, $21x = 1260$, and $21x + 324 = 1584$, as before. Here both ſides are equal to 1584, which is 12 times 132.

The fourth ſtep is derived from the third, by ſubtracting $21x$ or 1260 from both ſides of the equation. Now $3x = 180$, and $3x + 144 = 180 + 144 = 324$. Moreover, if 1260 be ſubtracted from 1584, the remainder will be 324.

In the fifth ſtep 144 is ſubtracted from both ſides of the equation. Now $3x = 180$, and if 144 be ſubtracted from 324, the remainder will be 180.

Laſtly, if $3x$ be divided by 3 the quotient is x; and if 180 be divided by 3 the quotient is 60; ſo both ſides of the equation are equal to 60, which is $\frac{1}{3}$d of 180, the value of both ſides of the equation in the ſtep preceding.

76. It is not to be expected we ſhould thus trace out the numerical value of every ſtep in every example of ſimple equations; but the doing of this may be uſeful to a learner.

77. Example 2. Let $\frac{3x}{5} - 1 = \frac{4x}{7}$

$$\frac{21x}{5} - 7 = 4x$$

$$21x - 35 = 20x$$

$$21x - 20x - 35 = 0,$$

that is, $x - 35 = 0$

$$x = 35.$$

Proof. $3x = 105$, $\frac{3x}{5} = 21$, $\frac{3x}{5} - 1 = 20$. Again, $4x = 140$, $\frac{4x}{7} = 20$, as before.

78. Ex-

78. Example 3. Let $\frac{8x}{5}-11=\frac{9x}{10}-4$.

$$8x-55=\frac{9x}{2}-20$$

$$16x-110=9x-40$$

$$16x-9x-110=-40,$$

that is, $7x-110=-40$

$$7x=-40+110=70$$

$$x=10.$$

Proof. $8x=80$; $\frac{8x}{5}=16$; $\frac{8x}{5}-11=5$. Again, $9x=90$; $\frac{9x}{10}=9$; $\frac{9x}{10}-4=5$, as before.

79. Example 4. Let $\frac{5x}{6}-9=\frac{10x}{11}-19$

$$5x-54=\frac{60x}{11}-114$$

$$55x-594=60x-1254,$$

$$-594=60x-55x-1254,$$

that is $-594=5x-1254,$

or rather $5x-1254=-594$

$$5x=1254-594=660$$

$$x=132.$$

Proof. $5x=660$; $\frac{5x}{6}=110$; $\frac{5x}{6}-9=101$. Again, $10x=1320$; $\frac{10x}{11}=120$; $\frac{10x}{11}-19=101$, as before.

80. Example 5. Let $\frac{29x}{35}-8=110-\frac{6x}{7}$

$$\frac{29x}{5}-56=770-6x$$

$$29x-280=3850-30x$$

$$59x-280=3850$$

$$59x=3850+280=4130$$

$$x=70$$

Proof. $29x=2030$; $\frac{29x}{35}=58$; $\frac{29x}{35}-8=50$. Again, $6x=420$; $\frac{6x}{7}=60$; and $110-60=50$, as before.

81. Example 6. Let $\frac{x}{4}-2=3-\frac{x}{6}$

$$x-8=12-\frac{4x}{6}$$
$$6x-48=72-4x$$
$$10x-48=72$$
$$10x=72+48=120$$
$$x=12.$$

Proof. $\frac{x}{4}=3$; and $\frac{x}{4}-2=1$. Again, $\frac{x}{6}=2$ and $3-\frac{x}{6}=1$, as before.

82. Example 7. Let $\frac{x}{2}-3=\frac{x}{6}+5$

$$x-6=\frac{x}{3}+10$$
$$3x-18=x+30$$
$$3x-x-18=+30,$$
that is $2x-18=30$
$$2x=30+18=48$$
$$x=24.$$

Proof. $\frac{x}{2}=12$; $\frac{x}{2}-3=9$. Again, $\frac{x}{6}=4$, and $\frac{x}{6}+5=9$, as before.

83. Example 8. Let $\frac{x}{6}-3=\frac{x}{2}+5$

$$\frac{x}{3}-6=x+10$$
$$x-18=3x+30$$
$$-18=3x-x+30,$$
that is $-18=2x+30,$

or rather $2x+30=-18$

$$2x = -18 - 30 = -48$$
$$x = -24.$$

Proof. $\frac{x}{6} = -4$, and $\frac{x}{6} - 3$, or $-4 - 3 = -7$.

Again, $\frac{x}{2} = -12$, and $\frac{x}{2} + 5 = -12 + 5 = -7$, as before.

84. *Remarks* on theſe two laſt examples.

In this laſt example we ſee, that if x be -24 the original equation is a true equation according to the rules of algebra; for both ſides are equal to -7. But it will be ſaid, What does this -24 and this -7 mean? The former example, where x was 24, and both ſides of the equation 9, was plain enough; but this -24 is perfectly unintelligible. And ſo indeed it is, if x ſtands for a number in the abſtract. Nothing is plainer, than that the half of any number is greater than a ſixth part of the ſame number, or that $\frac{x}{2}$ is greater than $\frac{x}{6}$. Now in the former example, it is affirmed in the original equation, that if 3 be ſubtracted from the greater of theſe two quantities, and 5 be added to the leſs, they will then be reduced to an equality; all which is very poſſible. But in the latter example it is affirmed (in the original equation) that if 3 be ſubtracted from the leſs of the two, and 5 be added to the greater, they will then be made equal; a thing utterly impoſſible and abſurd.

85. But if x dos not ſtand for number in the abſtract, but as applied to ſome ſubject (in its nature capable of the diſtinction into addititious and ſubductitious quantities) then the caſe may be altered. Thus, let x ſtand for the worth of ſome merchant, and let negative numbers ſtand for his debts; then the original equation, properly tranſlated, will run thus; $\frac{1}{6}$ of this man's worth, together with a debt of 3 pounds, is juſt as much in value as $\frac{1}{2}$ of his worth with a credit of 5 pounds. Now what is he worth? And the

anſwer is, He is 24 pounds worſe than nothing, or 24 pounds in debt; which is plain. For $\frac{1}{6}$ of his worth is a debt of 4 pounds, which together with a debt of 3 pounds, makes a debt of 7 pounds. Again, $\frac{1}{2}$ his worth is a debt of 12 pounds, which together with a credit of 5 pounds is in value the ſame as a debt of 7 pounds as before: therefore his original worth was rightly eſtimated at -24 pounds.

86. Example 9. Let $20-\frac{5x}{6}=21-\frac{7x}{8}$

$$120-5x=126-\frac{42x}{8}$$
$$960-40x=1008-42x$$
$$960+42x-40x=1008$$
$$960+2x=1008$$
$$2x=1008-960=48$$
$$x=24.$$

Proof. $5x=120$; $\frac{5x}{6}=20$; and $20-\frac{5x}{6}=0$. Again, $7x=168$, $\frac{7x}{8}=21$, and $21-\frac{7x}{8}=0$, as before.

87. Example 10. Let $\frac{4x}{5}=\frac{174-6x}{7}$

$$4x=\frac{870-30x}{7}$$
$$28x=870-30x$$
$$28x+30x=870$$
$$58x=870$$
$$x=15.$$

Proof. $4x=60$; $\frac{4x}{5}=12$. Again, $6x=90$; $174-6x=174-90=84$, and $\frac{174-6x}{7}=\frac{84}{7}=12$, as before.

88. Example 11. Let $\frac{2x}{3}+\frac{99-5x}{6}=15$

2x

$$2x+\frac{99-5x}{2}=45$$

$$4x+99-5x=90$$

$$-x+99=90,$$

that is $+x-99=-90$

$$x=99-90=9.$$

In the 5th step we change the sign of every quantity in the 4th step; in order that x may come out affirmative. For we are to find the value of $+x$, not of $-x$. That such a change may be made justly is evident, for it is transposing the whole equation.

Proof. $2x=18$; $\frac{2x}{3}=6$; $5x=45$; $99-5x=99-45$ $=54$; $\frac{99-5x}{6}=\frac{54}{6}=9$; therefore $\frac{2x}{3}+\frac{99-5x}{6}=6+9$ $=15$.

89. Example 12. Let $\frac{56}{5x+3}=\frac{63}{14x-5}$

$$56=\frac{315x+189}{14x-5}$$

$$784x-280=315x+189$$

$$784x-315x-280=189$$

$$469x-280=189$$

$$469x=189+280=469$$

$$x=1.$$

Proof. $5x=5$; $5x+3=8$; $\frac{56}{5x+3}=\frac{56}{8}=7$. Again, $14x=14$; $14x-5=9$; $\frac{63}{14x-5}=\frac{63}{9}=7$, as before.

90. Example 13. Let $\frac{60}{11x-1}=\frac{70}{13x-3}$

$$60=\frac{770x-70}{13x-3}$$

$$780x-180=770x-70$$

$$780x-770x-180=-70$$

$$10x-180=-70$$

$$10x = 180 - 70 = 110$$
$$x = 11.$$

Proof. $11x = 121$; $11x - 1 = 120$; $\frac{60}{11x - 1} = \frac{60}{120}$ $= \frac{1}{2}$. Again, $13x = 143$; $13x - 3 = 140$; and $\frac{70}{13x - 3}$ $= \frac{70}{140} = \frac{1}{2}$, as before.

91. Example 14. Let $\frac{49}{8x + 1} = \frac{70}{12x + 1}$

$$49 = \frac{560x + 70}{12x + 1}$$
$$588x + 49 = 560x + 70$$
$$588x - 560x + 49 = 70$$
$$28x + 49 = 70$$
$$28x = 70 - 49 = 21$$
$$x = \frac{21}{28} = \frac{3}{4}.$$

Proof. $8x = \frac{8 \times 3}{4} = \frac{24}{4} = 6$; $8x + 1 = 7$, and $\frac{49}{7} = 7$. Again, $12x = \frac{12 \times 3}{4} = \frac{36}{4} = 9$; $12x + 1 = 10$, and $\frac{70}{10}$ $= 7$, as before.

92. Example 15. Let $\frac{6x}{7} - \frac{103 - 3x}{8} = 13$.

Here it is evident, that the latter fraction $\frac{103 - 3x}{8}$ is to be ſubtracted from the former fraction $\frac{6x}{7}$, and that their difference is 13. The cleareſt way of carrying on the work is to put the *vinculum* over the latter fraction, and uſe the character for multiplication till you are quit of the denominators by proceſs 1. Thus

$$6x - 7 \times \overline{\frac{103 - 3x}{8}} = 91$$
$$48x - 7 \times \overline{103 - 3x} = 728,$$

But

But $\qquad 7 \times \overline{103-3x} = 721-21x$;
Now from $\qquad 48x$
Subtract $\qquad 721-21x$
There remains $\qquad -721+69x=728$:
Hence $\qquad 69x=728+721=1449$
And $\qquad x=21$.

Otherwise, you might at first have reduced the two fractions to others of the same value, having one common denominator, and then have subtracted the numerator of the latter fraction from the numerator of the former, by the rule for subtracting fractions, thus: $\frac{6x}{7}=\frac{48x}{56}$ and $\frac{103-3x}{8}=\frac{721-21x}{56}$. Subtract the latter from the former, and we have their difference equal to $\frac{-721+69x}{56}=13$ by the hypothesis, and $\qquad -721+69x=728$, as before.

Proof. $6x=126$; $\frac{6x}{7}=\frac{126}{7}=18$; $3x=63$; $103-3x=103-63=40$; $\frac{103-3x}{8}=\frac{40}{8}=5$; and $\frac{6x}{7}-\frac{103-3x}{8}=18-5=13$.

OF PROBLEMS.

93. In every problem there are what mathematicians call, the *data* and the *quæsita*; some conditions laid down, and some quantities sought. For instance, suppose it was asked, What two numbers are those whose sum is 20, and whose difference is 12? Here are two conditions laid down; first, that the two numbers added together make 20; secondly, that if the less number be subtracted from the greater, the remainder will be 12. There are also two unknown

quantities ſought, the greater of the two numbers and the leſs. Now we are to argue algebraically from the conditions laid down, and from thence infer what the two numbers are.

94. It muſt be obſerved, that the number of *independent* conditions ought to be the ſame with the number of unknown quantities. If the number of conditions be fewer, the problem will not be limited to one anſwer. For inſtance, if it was aſked, What two numbers are thoſe whoſe ſum is 20? Here it is plain there are many anſwers in whole numbers, and innumerable, if fractions be admitted. If there are more conditions than unknown quantities, they are at leaſt ſuperfluous, and may be inconſiſtent. The conditions here ſpoken of muſt be *independent* of each other. All ſuch conditions as can be derived from each other are to be counted as only one condition. Thus, if it be aſked, What two numbers are thoſe whoſe ſum is 20, and twice whoſe ſum is 40? Here the latter condition is implied in the former. You learn nothing more by being informed that twice their ſum is 40, than what you would have known without ſuch information.

95. It is very uſeful when a problem is propoſed to conſider whether the *data* and *quæſita* do thus anſwer. Sometimes the number of the *data* will be clear, that of the *quæſita* doubtful; ſometimes the number of the *quæſita* be clear, and the *data* obſcure. But the conſideration of one will help to clear up the other, and make the circumſtances of the problem better underſtood.

96. In the ſolution, you are to ſubſtitute a letter arbitrarily for one of the unknown quantities; and to infer from one of the conditions an algebraic expreſſion or name for the other unknown quantity. Of courſe there will be left the other condition, from which you have not as yet argued at all. When you have thus got an algebraic expreſſion or name for each of the two unknown quantities; tranſlate this re-

maining

maining condition into algebraic language, which will furnish you with an algebraic equation. This equation ſolved by the methods before laid down will give the value of the unknown quantity ſought.

97. The ſame condition muſt not be uſed twice; it will only lead to an identical propoſition, and therefore futile.

98. If the nature of the problem is ſuch as ſhows that one of the unknown quantities is greater than the other, ſubſtitute for the leaſt, and find an expreſſion for the other from one of the conditions of the problem.

99. We have here ſuppoſed that there is not more than two unknown quantities ſought. The directions here given hold good *(mutatis mutandis)* whatever be the number of unknown quantities. But it is of very little uſe to ſpend time about the ſolution of problems in which many unknown quantities are concerned.

We ſhall illuſtrate the precepts here laid down in the ſolution of the problem before propoſed.

100. Now it muſt be obſerved, that when there are two conditions, and alſo two unknown quantities, there will of courſe be four methods of ſolution. *Firſt*; You may ſubſtitute for the greater of the two unknown quantities, and either from the firſt condition infer an expreſſion for the leſs unknown quantity, and from the ſecond condition get an equation; or, from the ſecond condition infer an expreſſion for the leſs unknown quantity, and from the firſt condition draw your equation. But, *ſecondly*; You may ſubſtitute for the leſs of the two unknown quantities, and either from the firſt condition infer an expreſſion for the greater unknown quantity, and from the ſecond condition get an equation; or, from the ſecond condition infer an expreſſion for the greater unknown quantity, and from the firſt condition draw your equation. We ſhall purſue the ſolution of this one

pro-

problem, through each of these methods, as an example once for all.

101. PROBLEM 1. Given the sum of two numbers 20, and their difference 12. What are those two numbers?

METHOD 1st. Substitute for the greater number x. Now the first condition is, that the two numbers added together make 20. Whence I infer, that if either of those numbers (the greater for instance) be taken from 20, the remainder will be the other; that is (in this instance) the less. Take therefore the greater number x, from 20, and the remainder or $20-x$ will be the less. Having thus got an expression for the greater number by an arbitrary substitution, and for the less, by inference from the first condition, subtract the latter, (or $20-x$) from the former (or x) and the remainder (or $-20+2x$) is their difference, which by the second condition is 12: which equation solved gives the value of x. Thus:

The greater number	x
The less	$20-x$
Their difference	$-20+2x$
Whence this equation	$-20+2x=12$
By process 3d	$2x=12+20=32$
By process 4th	$x=16$.

Having thus found the value of x, the greater of the two quantities, substitute that known value for x in the second step, and you have $20-16$ or 4 for the less.

102. METHOD 2d. Substitute as before for the greater number x. Now the second condition is, that their difference (or the excess of the greater number above the less) is 12; therefore if 12 be taken from the greater it will leave the less; subtract therefore 12 from x, and the remainder (or $x-12$) is the less. Now, having an expression for each of the two numbers, translate the first condition into algebraic language, and you have an equation. Thus:

The

The greater number	x
The lefs	$x-12$
Their fum	$2x-12$
Whence this equation	$2x-12=20$
	$2x=20+12=32.$
	$x=16$, as before.

The lefs number is got by putting for x (in the fecond ftep) its value now found; thus $16-12=4$, as before.

103. METHOD 3d. Let us now change the fubftitution, and put y for the lefs of the two unknown quantities; but draw the expreffion for the greater from the firft condition. We before fhewed, that if the two numbers added together make 20, then if either of them be fubtracted from 20, the remainder will be the other. Take therefore the lefs number y from 20, and the remainder or $20-y$ will be the greater. Having thus got an expreffion for each of the two unknown quantities, tranflate the fecond condition and we have an equation. Thus:

The lefs number	y
The greater	$20-y$
Their difference	$20-2y$
Whence this equation	$20-2y=12$
	$-2y=12-20$
Or rather (fee par. 88.)	$2y=-12+20=8$
	$y=4.$

Having found the value of y, fubftitute this value for y in the fecond ftep, and we have $20-4=16$ the greater number, as before.

104. METHOD 4th. Subftitute as before for the lefs number y, and the fecond condition is, that the excefs of the greater number above the lefs is 12. If then we add this excefs to the lefs number, we have the greater; therefore to y add 12, and the fum (or $y+12$) is the greater. Having an expreffion for each of the two numbers, tranflate the firft condition, and we have an equation. Thus:

The leſs number y
The greater number $y+12$
Their ſum $2y+12$
Whence this equation $2y+12=20$

$$2y=20-12=8$$
$$y=4.$$

Subſtitute for y in the ſecond ſtep its value now found, and we have $4+12=16=$ the greater number, as before.

The laſt method is the beſt, becauſe addition is eaſier than ſubtraction.

105 Here it may be obſerved, that if the ſame condition be uſed twice, we ſhall come to an identical propoſition. In the laſt caſe, the leſs number was y, the greater $y+12$, which expreſſion was got from the ſecond condition, namely, that the difference of the two numbers was 12. Let us try to uſe this condition again to get an equation. Therefore to find their difference, ſubtract the leſs y, from the greater $y+12$, and there remains 12, which by the ſame condition is equal to 12. We therefore come to this concluſion, that $12=12$, or that 12 is 12! The caſe would have been the ſame in any of the other four methods.

106. We have purſued this problem through all four methods. It is ſufficient to uſe one method; and in problems having only one unknown quantity, there is but one method.

107. *A problem of the ſame kind.*

The number of freeholders that voted at a certain election was 1296; the ſucceſsful candidate carried it by a majority of 120. How many voted on each ſide?

108. Problem 2. Three perſons, A, B, and C, make a joint contribution, which in the whole amounts to 100 pounds. Of this A contributes a ſum unknown. B twice as much as A and 10 pounds more. C as much as A and B together. I demand their ſeveral contributions,

Solution. Let A's ſhare $= x$
B's ſhare $=2x+10$
C's ſhare $=3x+10$

Equa-

$$\text{Equation} \quad 6x + 20 = 100$$
$$6x = 100 - 20 = 80$$
$$x = 13\tfrac{1}{3} = 13\ \pounds.\ 6\ s.\ 8\ D.$$

		£.	S.	D.
Proof.	*A*'s ſhare	=13	6	8
	B's ſhare	=36	13	4
	C's ſhare	=50	0	0
	Total	100	0	0.

Remarks. In this problem there are plainly three unknown quantities, namely, the three ſums contributed; and there are alſo three conditions. Firſt, that *B*'s ſhare was twice as much as *A*'s, and 10 pounds more. Secondly, that *C*'s ſhare was as much as both *B*'s and *A*'s together. Thirdly, that all the three ſhares together amounted to 100 pounds. It is evident from the terms of the problem, that *A*'s ſhare is the leaſt, and therefore, according to the precept before given, we ſubſtitute for *A*'s ſhare, and get an expreſſion for the other two from the terms of the problem.

109. PROBLEM 3. One goes with a certain quantity of money about him to a tavern, where he borrows as much as he had then about him, and out of the whole ſpends a ſhilling; with the remainder he goes to a ſecond tavern, where he borrows as much as he had then left, and there alſo ſpends a ſhilling; and ſo he goes on to a third and a fourth tavern, borrowing and ſpending as before, after which he had nothing left. I demand how much money he had at firſt about him?

110. In this problem, it is evident there is only one unknown quantity, namely, the money he had at firſt about him. Of courſe there is only one condition, although at firſt ſight it ſeems as if there were many. A long train of circumſtances are related; ſometimes increaſing, ſometimes decreaſing the money in his purſe. But the condition laid down is, that after all the

the changes before described, "he had nothing left." In other words, that what he spent, was equal to his original money, together with all that he borrowed.

111. As there is only one unknown quantity, there can be no room for choice in the substitution. Let x then stand for his original money, which may be considered either as a certain number of shillings, or a certain number of pence. In the first case we must call the money he spends at each tavern one shilling, and we shall have the answer in shillings. In the latter case we must call the money he spends at each tavern twelve-pence, and the answer will come out in pence. And we may here remark, once for all, that all quantities of the same kind, must be expressed by *one and the same denomination*, in which denomination the answer will be given.

112. That we may distinctly trace all the changes made in his money, it will be proper to attend him with a pocket-book, and enter his account as he goes on, in manner following:

Original money - - -	x
Borrowed at the 1st tavern -	x
Money after borrowing -	$2x$
Spent at the 1st tavern -	$12d.$
Money left to go to the 2d tavern	$2x-12$
Borrowed at the 2d tavern -	$2x-12$
Money after borrowing -	$4x-24$
Spent at the 2d tavern -	12
Money left to go to the 3d tavern	$4x-36$
Borrowed at the 3d tavern -	$4x-36$
Money after borrowing -	$8x-72$
Spent at the 3d tavern -	12
Money left to go to the 4th tavern	$8x-84$
Borrowed at the 4th tavern -	$8x-84$
Money after borrowing -	$16x-168$
Spent at the 4th tavern -	12
Money (finally) left -	$16x-180$

Hence

Hence this equation - $16x-180=0$
Therefore - $16x=180$

$\overset{D.}{x=11\frac{1}{4}}$, as will appear, by keeping the account as above.

113. PROBLEM 4. A man and his wife and child dine together at an inn. The landlord charged 10 pence for the child. He charged for the woman, as much as for the child, and one third of what he charged for the man. But for the man he charged as much as for the woman and child together. What did he charge for each?

For what the man paid substitute - x

Then the woman paid (by the 1st condition) $10+\frac{x}{3}$

The man paid (by the 2d con.) $10+10+\frac{x}{3}$ or $20+\frac{x}{3}$

Therefore - $20+\frac{x}{3}=x$.

Equation - $20+\frac{x}{3}=x$.

$60+x=3x$

$60=2x$

$D.$

$30=x$, the man's share.

Whence $10+\frac{x}{3}$, or the woman's share, $=10+10\overset{D.}{=}20$.

Otherwise. For the woman's share put y, and from the second condition, the man's share is $y+10$, but by the first condition, $y=10+\frac{y+10}{3}$, whence $3y=30+y+10$ or $3y=y+40$, whence $2y=40$, and $y=20$, as before.

114. *A problem of the same kind.*

Three persons subscribe to a charity. *A* subscribes 100 pounds, *B* subscribes as much as *A* and one third of what *C* subscribes, *C* subscribes as much as *A* and half *B*. What did each subscribe?

115. PRO-

115. PROBLEM 5. A man leaves his ſortune, which was 560 pounds, between his ſon and daughter, in this manner: For every half crown the ſon ſhould have, the daughter was to have a ſhilling. What were their reſpective fortunes?

Put x for the ſon's fortune in pounds, then $8x$ is the number of half crowns he had, therefore $8x$ is alſo the number of ſhillings the daughter had; and $\frac{8x}{20}$ her fortune in pounds: whence $x+\frac{8x}{20}=560$ pounds, the father's fortune, and we have this equation:

$$x+\frac{8x}{20}=560$$

$$20x+8x=11200$$

or $$28x=11200$$

and $$x=400 \text{ pounds.}$$

The daughter's fortune 160 pounds.

116. PROBLEM 6 *. Divide 560 into two ſuch parts, that one part may be to the other as 5 to 2.

Put x for one part, and then $560-x$ is the other. But as 5 is to 2, ſo is x to $\frac{2x}{5}$, (by the rule of three) therefore $\frac{2x}{5}$ is alſo the other part, therefore equal to $560-x$, that is $\frac{2x}{5}=560-x$

whence $$2x=2800-5x$$

and $$7x=2800$$

$$x=400$$

$$560-x, \text{ or the other part, is } 160.$$

It is manifeſt that this laſt problem coincides with the former, and that the former may be ſolved on this principle; that the father's fortune muſt be ſo divided between the two children that their ſhares may

* Perhaps it may be proper to poſtpone this problem till after par. 126. It is placed here becauſe of its relation to the foregoing problem.

be

be to each other as half a crown to a ſhilling, or as 5 ſix-pences to 2 ſix-pences; or as 5 to 2.

117. PROBLEM 7. Divide 20 into two ſuch parts that 3 times one part added to 5 times the other may make 74.

For one part put	x
For the other part	$20-x$
Three times one part	$3x$
Five times the other part	$100-5x$.

Hence $3x+100-5x=74$, or $100-2x=74$

$$\therefore 100-74=2x=26$$

$$x=13$$

And the other part is 7.

There is nothing immediately diſcoverable in the conditions of this problem, that determines one of the two unknown parts to be bigger than the other, and x may ſtand for either of them. But in the third ſtep, x is plainly conſidered as ſtanding for that part which is to be taken 3 times, whether that in the iſſue proves to be the greater or leſs part, and therefore in the ſubſequent work, x is reſtrained to mean that part, and that only.

118. PROBLEM 8. Two perſons *A* and *B* engage at play. *A* has 72 guineas, and *B* 52, before they begin. After a certain number of games won and loſt between them, *A* riſes with three times as many guineas as *B*. I demand how many guineas *A won* of *B*?

For the number of guineas *A* won put	x
Then *A* riſes with	$72+x$
B riſes with	$52-x$
But by the problem	$72+x=3\times\overline{52-x}$
Whence	$x=21$.

119. In many caſes the ſolution of a problem is made eaſier by ſeeking, not the unknown quantity required, but ſome other unknown quantity, from which the former may be derived. Thus, inſtead of ſeeking the money *A* won of *B*, let us enquire what each riſes

with, and thence infer what *A* won. Put therefore y for the money *B* riſes with, and the money *A* riſes with is of courſe $3y$. Now, as what one loſes the other wins, they muſt have the ſame money between them at riſing up, as at ſitting down; that is, $3y+y=72+52$ or $4y=124$, and $y=31$. Whence it follows that *B* has loſt 21 guineas, of courſe *A* has *won* 21 guineas, which is a direct anſwer to the demand in the problem.

120. PROBLEM 9. A ſtudent coming into a bookſeller's ſhop, aſks the price of his ſtock of books; to which the bookſeller replies, I will have four ſhillings a volume for them, one with another. No, replies the ſtudent, I have not money enough by five pounds to pay for them at that rate, but if you will let me have them at three ſhillings and four pence a volume, I can then pay for the whole and ſhall have five pounds left to pay for the carriage. What number of volumes had the bookſeller, and how much money had the ſtudent?

121. There are in this problem ſeemingly two unknown quantities, the number of volumes in the ſhop, and the money the ſtudent had in his pocket. But theſe are dependent on each other: if you know the one, you can infer the other from it. Subſtitute for either, for inſtance, for the number of volumes, and thence compute the ſtudent's money, on the firſt ſuppoſition, that the books were four ſhillings a volume. Again, compute the ſtudent's money, on the ſecond ſuppoſition, that the books were three ſhillings and four pence a volume, and you will have two different expreſſions for the ſtudent's money, which are therefore equal to each other.

122. Number of volumes - x

Their value (in ſhillings) on the firſt ſuppoſition $=4x$

Student's money (in ſhillings) - $=4x-100$

Value of the volumes (in ſhillings) on the ſecond ſuppoſition is $x\times3\frac{1}{3}=\frac{10x}{3}$

Stu-

Student's money (in ſhillings) on the ſecond ſuppoſition $\frac{10x}{3}+100$

Equation $4x-100=\frac{10x}{3}+100$.

Whence $x=300$ volumes, and the ſtudent had 1100 ſhillings or 55 pounds. The value of the books is 60 pounds at 4 ſhillings a volume, and 50 pounds at 3 ſhillings and 4 pence a volume.

123. PROBLEM 10. Given the difference of two numbers 10, and the difference of their ſquares 120, to find the numbers.

Let the leſs number be $=x$

Then from the firſt condition, the greater is $x+10$*

The ſquare of the greater $xx+20x+100$

The ſquare of the leſs xx

The difference of their ſq. $20x+100=120$

Whence $x=1$ the leſs, and $x+10$ or the greater $=11$.

124. *A problem without a ſolution.*

From each of 16 pieces of gold, an artiſt filed the worth of half a crown, and then offered them in payment for their original value. But being detected and the pieces weighed, they were found to be worth, in the whole, no more than 8 guineas. What was the original value of each piece?

125. PROBLEM 11. A perſon was hired to work for a year at 7 ſhillings a day, on this condition, that for every day he played, he ſhould forfeit 3 ſhillings. At the end of the year he had nothing to receive or to pay. How many days did he work?

For the number of days he worked put x

Then the number of days he played was $365-x$

Money earned $=7x$

Money forfeited $3\times\overline{365-x}=1095-3x$

Equation $7x=1095-3x$

* Here the firſt condition readily affords an expreſſion for the other unknown quantity; not ſo the ſecond condition.

 Whence

Whence $x = 109\frac{1}{2}$, that is, he worked 109 days and an half, and therefore played 255 days and an half.

126. Of the character for proportionality, and of turning *Analogies* into *Equations*, and *vice versa*.

When four numbers are of such a sort, that the first has the same proportion to the second, that the third has to the fourth, then such numbers are said to be proportional. Such numbers constitute the terms of what is called an *analogy*, and if the three first terms are given, the fourth may be found, by multiplying the second and third terms together, and dividing that product by the first term.

127. The character whereby such proportionality is expressed, is two dots interposed between the terms of the first pair, also two dots interposed between the terms of the second pair; but between the two intire pairs, four dots are interposed: thus, $3 : 4 :: 12 : 16$, and it is read thus, 3 is to 4 as 12 is to 16. So again, $a : b :: c : d$, that is, a is to b as c is to d. And in this case the fourth term $d = \frac{b \times c}{a}$.

128. Hence it follows, that $a \times d = b \times c$; that is, the product of the multiplication of the two extreme terms, is equal to the product of the multiplication of the two mean or middle terms. Whenever therefore the conditions of a problem furnish us with an analogy, that analogy implies an equation.

129. And if an equation be proposed, in which each side consists of the multiplication of two quantities; that equation can be turned into an analogy. For the two factors on one side of the equation are the two extreme terms; and the two factors on the other side, the two middle terms of the analogy. Thus, if $ax = bc$, then $a : b :: c : x$. Again, if $ax = b$, then $a : b :: 1 : x$, because b may be considered as $1 \times b$.

130. Problem 12. What number is that, which being severally added to 36 and 52, will make the former sum to the latter as 3 is to 4?

Let

Let the unknown number be x
Then the former ſum is $36+x$
the latter ſum is $52+x$

But by the terms of the queſtion we have this analogy; $36+x : 52+x :: 3 : 4$. which analogy gives this equation $\overline{36+x} \times 4 = \overline{52+x} \times 3$, that is, $144+4x = 156+3x$.

Whence $x=12$. For $36+12=48$, and $52+12=64$; now $48 : 64 :: 3 : 4$, becauſe $4 \times 48 = 3 \times 64$, each ſide being 192.

131. *Caſe* 2. What number is that, which being ſeverally added to 36 and 52, will make the former ſum to the latter, as 2 is to 3?

Here, as before, the former ſum is $36+x$
the latter ſum is $52+x$

Analogy $36+x : 52+x :: 2 : 3$, whence $\overline{36+x} \times 3 = \overline{52+x} \times 2$, or $108+3x=104+2x$ and $x=-4$.

-4 therefore is the number to be algebraically added to each of the two propoſed numbers, that their (algebraic) ſums may be to each other as 2 is to 3; but the algebraic addition of -4 is the ſame as the arithmetical ſubtraction of $+4$. That is, if we take the words of the problem in the vulgar and narrow ſenſe of *adding* arithmetically, it is impoſſible to find ſuch a number as the problem requires. But if we take the words in the enlarged algebraic ſenſe, and by *adding*, mean either an arithmetical adding, or an arithmetical ſubtracting, as the caſe may require; then the problem admits of an anſwer (ſee par. 23–26). And we are given to underſtand, that the conditions of the problem cannot be anſwered by the arithmetical addition of any number whatever, but it requires the arithmetical ſubtraction of a certain number, and that number is 4. And this appears plainly, for $36-4=32$, and $52-4=48$. Now $32 : 48 :: 2 : 3$, for $3 \times 32 = 2 \times 48$, each ſide being 96.

132. However confined the terms of the problem may be, as verbally expreſſed; when the conditions

are put into the algebraic character, they will then have a more extenſive ſenſe; and all that latitude of meaning, that is peculiar to the algebraic language.

133. *Caſe* 3. Let us find that number, which being ſeverally added to 36 and 52, will make the former ſum to the latter as 2 to 1.

Purſuing the ſteps of the former caſes, we ſhall have $x = -68$. Now if -68 be added (algebraically) to 36 and 52 ſeverally, the former ſum will be -32, the latter -16, which is the ſame, *as to proportion*, as $+32$ and $+16$; the diſtinction of affirmative and negative having no place in caſes of proportion.

The problem therefore in the enlarged algebraic ſenſe, is this; What number is that, which being ſeverally *incorporated* with 36 and with 52, will make two other numbers having the proportion aſſigned in the problem? In the firſt inſtance, the incorporation was performed by adding that number to the two given numbers: in the ſecond inſtance, it was performed by ſubtracting the number ſought from each of the two given numbers. But in the laſt inſtance, this incorporation was performed by ſubtracting the two given numbers from the number ſought.

134. PROBLEM 13. A footman who contracted for 8 pounds a year and a livery, was turned away at the end of 7 months, and received for his wages 2£. 13*s*. 4*d*. and his livery. What was the value of the livery?

For the value of the livery put x, then his year's wages is $8+x$. To find his wages for 7 months, ſay, if 12 months give $8+x$, what will 7 months give? and the anſwer is $\frac{56+7x}{12}$: but he received $£.2\frac{2}{3}+x$; therefore we have this equation $\frac{56+7x}{12} = 2\frac{2}{3}+x = \frac{8}{3}+x$. Multiply both ſides by 3, and we have $\frac{56+7x}{4}$ $=8+$

$=8+3x$, whence $56+7x=32+12x$, and $56=32+12x-7x=32+5x$, hence $5x=24$ pounds and $x=4\frac{4}{5}$ pounds $=4$ pounds and 16 ſhillings.

For his year's wages is worth 12£. 16*s*. and 7 month's wages is worth 7£. 9*s*. 4*d*. (at a guinea and a groat a month) from this take 2£. 13*s*. 4*d*. and there remains 4£. 16*s*. the value of the livery as before.

135. Problem 14. A ſhepherd driving a flock of ſheep in time of war, meets with a company of ſoldiers, who plunder him of half his flock and half a ſheep over. The ſame treatment he meets with from a ſecond, a third, and a fourth company; every ſucceeding company plundering him of half the flock the laſt had left and half a ſheep over, inſomuch that at laſt he had only eight ſheep left. I demand how many he had at firſt?

For the number he had at firſt put x.

Taken by the firſt company $\frac{x}{2}+\frac{1}{2}=\frac{x+1}{2}$

Subtract this from x, or $\frac{2x}{2}$, and there remains $\frac{x-1}{2}$

Taken by the 2d company $\frac{x-1}{4}+\frac{1}{2}=\frac{x-1}{4}+\frac{2}{4}=\frac{x+1}{4}$

Subtract this from $\frac{x-1}{2}$ or $\frac{2x-2}{4}$, there remains $\frac{x-3}{4}$

Taken by the 3d company $\frac{x-3}{8}+\frac{1}{2}=\frac{x-3}{8}+\frac{4}{8}=\frac{x+1}{8}$

Subtract this from $\frac{x-3}{4}$ or $\frac{2x-6}{8}$, there remains $\frac{x-7}{8}$

Taken by the laſt company $\frac{x-7}{16}+\frac{1}{2}=\frac{x-7}{16}+\frac{8}{16}=\frac{x+1}{16}$

Subtract this from $\frac{x-7}{8}$ or $\frac{2x-14}{16}$, there remains $\frac{x-15}{16}$ left by the laſt company. Therefore $\frac{x-15}{16}=8$, and $x=143$.

136. This queſtion has the appearance of ſomething paradoxical; for how could the ſoldiers take half a ſheep? But if the original flock was an odd number, then half the flock conſiſts of an integral number and half a ſheep beſides, and half the flock with half a ſheep over is an integral number of ſheep. If the remainder be an odd number, the caſe will be the ſame reſpecting the ſecond company. And ſo we ſee it happens, with reſpect to all four companies; but if a fifth was to come, the ſame paradoxical circumſtance could not take place.

137. Problem 15. One buys a certain number of eggs, half whereof he buys at two a penny, the other half at three a penny. Theſe he afterwards ſold out again at the rate of five for two pence, and contrary to his expectation loſt a penny by the bargain. What was the number of eggs?

Subſtitute for the number of eggs $= x$.

Value of half his eggs or $\frac{x}{2}$ at 2 a penny $= \frac{x}{4}$

Value of half his eggs or $\frac{x}{2}$ at 3 a penny $= \frac{x}{6}$

Paid for the whole $\frac{x}{4} + \frac{x}{6} = \frac{5x}{12}$.

Value of the whole number at 5 for two pence is $\frac{2x}{5}$, whence $\frac{5x}{12} - \frac{2x}{5} = 1$, or $\frac{25x - 24x}{60} = 1$, or $\frac{x}{60} = 1$, and $x = 60$.

138. This queſtion has alſo the air of a paradox. The jingle of the words miſleads. Had he bought one half, at one egg for two pence, and the other half at one egg for three pence; then he might have ſold the whole at two eggs for five pence without loſs. There being an equal number of the cheaper and of the dearer eggs, he might have taken an egg out of one half and an egg out of the other half to pair with it, and ſold theſe two for five pence without loſs. But if he attempts thus to lot each half in our caſe,

taking

taking two eggs out of one moiety, and three eggs out of the other moiety; the latter moiety (the cheaper ſort) will fail, when he has made up ten lots, and there will remain ten eggs of the dearer ſort, and none left of the cheaper ſort to pair with them. Theſe ten coſt him five pence. Now if he ſells them alſo at five for two pence, they will bring him only four pence; ſo that he will loſe one penny by his bargain.

139. PROBLEM 16. A courier paſſing through a certain place *A*, travels at the rate of five miles in two hours. Four hours after another paſſes through the ſame place, travelling the ſame way at the rate of ſeven miles in two hours. I demand how long, and how far the firſt muſt travel before he is overtaken by the ſecond?

140. Before we ſet about the ſolution of this problem, it may be proper to remark, that there is only one unknown quantity, and only one condition. For though it is aſked not only how long, but likewiſe how far the firſt travelled, yet theſe two circumſtances are dependent on each other; the diſtance travelled can be inferred from the time, or the time from the diſtance; as the rate of travelling is given in the terms of the problem. The condition (and the only condition) is, that the latter overtakes the former, which implies that they are then together; and of conſequence each at the ſame diſtance from the place *A*; this equality of their reſpective diſtances from *A* furniſhes the equation.

141. Let x be the number of hours the firſt travelled, after he paſſed through *A*, and before he was overtaken; then will $x-4$ be the number of hours the ſecond travelled. To find how far the firſt travelled in the time x, ſay, if two hours carry him five miles, how far will x hours carry him? The anſwer is $\frac{5x}{2}$ miles. Again, to find how far the ſecond travelled in $x-4$ hours, ſay, if two hours carry him 7 miles, how far will $x-4$ carry him? The anſwer is

7x—

$\frac{7x-28}{2}$ miles. But when the ſecond overtakes the firſt, they will both be at the ſame diſtance from *A*, and both have travelled the ſame number of miles. Therefore $\frac{5x}{2}=\frac{7x-28}{2}$; whence $5x=7x-28$, and $x=14$ hours, the time the firſt travelled, and $x-4$ or $14-4=10$ hours, the time the ſecond travelled.

142. To find how far the firſt travelled, ſay, if two hours carry him 5 miles, how far will 14 hours carry him, and the anſwer is 35 miles.

Again, To find how far the ſecond travelled, ſay, if two hours carry him 7 miles, how far will 10 hours carry him, and the anſwer is 35 miles; which ſhews that he has overtaken the firſt.

143. *Caſe* 2. All things remaining as before, let now the ſecond courier travel at the rate of three miles in two hours, I demand when he will overtake the firſt in that caſe?

Put x for the time travelled by the firſt as before, and $x-4$ will be the time travelled by the ſecond. The miles travelled by the firſt will be $\frac{5x}{2}$, and the miles travelled by the ſecond will be $\frac{3x-12}{2}$, and we have this equation $\frac{5x}{2}=\frac{3x-12}{2}$; whence $x=-6$ hours, the time the firſt travelled, and $x-4=-6-4=-10$, the time the ſecond travelled. Again, if 2 hours carried the firſt 5 miles, how far will -6 hours carry him, and the anſwer is -15 miles. Likewiſe if 2 hours carry the ſecond 3 miles, how far will -10 hours carry him? and the anſwer is alſo -15 miles.

144. Here it may be aſked, what is the meaning of -6 hours, and -10 hours, and of -15 miles, and what is the real anſwer to the problem?

To anſwer this we muſt obſerve, that in ſolving the problem, the equation was derived from this circumſtance,

ſtance, that their reſpective diſtances from the place *A* were equal. Therefore the algebraic language made uſe of in the ſolution of the problem is more extenſive than the words in which it is propoſed. The enquiry in the algebraic ſolution is, at what time, and in what part of the road, the two couriers either *will be*, or *have been* together? whether that place be in the part of the road *to* which they are going, or from which they *have* come? ſince in either caſe, when together, they will both be at the ſame diſtance from the place *A*. Now, as all that part of the road from *A* forward, to which they are going, is called affirmative; ſo all that part of the road on the other ſide *A* (in the contrary direction) muſt be called negative. In like manner, as all the time to come after their paſſing through *A*, is called affirmative, ſo all the time paſt before their arrival at *A* muſt be called negative: future time being counted affirmative; paſt time muſt be counted negative. The meaning of the anſwer is, that the two couriers *were* together 15 miles from *A*, on the other ſide, from whence they came; and this happened ſix hours *before* the firſt, and ten hours *before* the ſecond paſſed through *A*. Nor muſt we be ſurpriſed to find an anſwer to a queſtion that was not aſked in the words of the problem, ſince the queſtion was plainly ſuppoſed in the algebraic equation.

145. It ſhould be obſerved, that this queſtion always admits of one anſwer, and one only. When the ſolution gives an affirmative anſwer, there is no negative one; and contrariwiſe. There is but one place in which the two couriers can be together. Before they arrive at that point, the ſwifter of the two, gaining ground on the ſlower, they will continually get nearer and nearer, till at laſt the ſwifter overtakes the ſlower. After that, the ſwifter outgoing the ſlower, they will ſeparate more and more, and never be together again. In the firſt caſe of this problem, the place and time when they were together happened *after* they paſſed through *A*; in the ſecond caſe it hap-

happened *before* they arrived at *A*. In this ſecond caſe the problem is *literally* impoſſible. For the ſecond courier, not only travelling ſlower than the firſt, but likewiſe ſetting out later, can never *overtake* him.

146. The conditions of the problem may be ſuch that it is not capable of an anſwer; no, not in that extenſive ſenſe which the algebraic characters admit of. A problem may be ſo propoſed as to be incapable either of an affirmative or a negative anſwer; and yet rules are laid down which, one would think, muſt mechanically produce an anſwer. And if we work by thoſe rules, ſhall we not get an anſwer to a problem, which cannot poſſibly have an anſwer? And if ſo, what muſt we think of ſuch rules?—We ſhall anſwer this hereafter.

147. When two unknown quantities are concerned in a problem, we are to ſubſtitute a letter for one of them, and deduce an expreſſion for the other from one of the conditions of the problem, according to the directions before given. But the ſolution of the problem will be more *technical*, and therefore eaſier, if letters be ſubſtituted for every one of the unknown quantities. Having by this means got an expreſſion in the algebraic language for every unknown quantity that occurs in the problem, the conditions will be readily tranſlated into that language. The conditions thus tranſlated, are called *fundamental equations*. If theſe fundamental equations contain fractional quantities, they muſt be freed from fractions by the firſt proceſs formerly laid down. Hence will be derived two others called ſecondary equations, which are to be numbered, and called the firſt and ſecond equations: all the following equations, derived from them, are alſo to be numbered, beginning with theſe two. By means of the firſt and ſecond equations, compared together, one of the unknown quantities muſt be exterminated; ſo that from thence may reſult an equation containing one unknown quantity only, to be ſolved by the rules already laid down.

148. There

148. There are three methods of exterminating any one of the unknown quantities by means of a pair of equations. The first is by deriving from the two secondary equations two others; in each of which one of the unknown quantities shall be found with the same co-efficient. For then adding those two equations, when the unknown quantity has unlike signs, or subtracting one equation from the other, when the unknown quantity has like signs, will give an equation in which that unknown quantity is wanting. Now two such equations, may be derived from the secondary equations, by multiplying the first equation by the co-efficient of the unknown quantity in the second equation, and again multiplying the second equation by the co-efficient of the unknown quantity in the first equation. And here it may be observed, that if the co-efficient of one of the unknown quantities in either equation is an aliquot part of the co-efficient of the same unknown quantity in the other equation, we may multiply the former equation by that aliquot part, and we shall have two equations in which that unknown quantity is found with the same co-efficient in each; and that by virtue of one multiplication only.

149. The second method is to find from each of the two secondary equations, a value of one of the unknown quantities, in terms in which the other only is concerned, and then equating these two values to one another.

150. The third method is to find a value of one of the unknown quantities in terms in which the other only is concerned, from one of the secondary equations, as before, and then substituting that value for the same unknown quantity in the other secondary equation.

We shall give the solution of the following problem in each of these three methods, to illustrate what has been said.

151. PROBLEM 17. It is required to find two num-

numbers with the following properties, That half the first with a third part of the second may make 16, and that a fourth part of the first with a fifth part of the second may make 9.

Let the first number be x, the second y.

Fundamental equations $\begin{cases} \frac{x}{2}+\frac{y}{3}=16 \\ \frac{x}{4}+\frac{y}{5}=9 \end{cases}$

Secondary equations $\begin{cases} \text{Eq. 1.} & 3x+2y=96 \\ \text{Eq. 2.} & 5x+4y=180. \end{cases}$

Whence by the *first method* we have

Equat. 3. $15x+10y=480$

Equat. 4. $15x+12y=540.$

Subtract equat. 3 from equat. 4. and we have

Equat. 5. $* \ +2y=60$

and Equat. 6. $y=30.$

To find the value of x, substitute for y in the first equation its value now found, and $3x+60=96$, whence $x=12$.

152. Let us now exterminate y, by this same first method. This is rather to be chosen, because the coefficient of y in the first equation is an aliquot part of the co-efficient of y in the second equation; multiply equation the first by 2, and we have

Equat. 3. $6x+4y=192.$

Subtract equat. 2. from equat. 3. and we have

Equat. 4. $x=12.$

The value of y will be found from either of the secondary equations, as before.

153. Let us now exterminate x, by the *second method*.

From equat. 1. we have equat. 3. $x=\frac{96-2y}{3}$

From equat. 2. we have equat. 4. $x=\frac{180-4y}{5}$:

Hence equat. 5. $\frac{96-2y}{3}=\frac{180-4y}{5}$.

This

This equation contains only one unknown quantity, and therefore may be ſolved by the rules before laid down, and gives in the concluſion $y=30$.

154. Let us now exterminate x, by the *third method*. From equation 1. we have (as in the ſecond method)

Equat. 3. $x=\frac{96-2y}{3}$,

Whence equat. 4. $5x=\frac{480-10y}{3}$,

Subſtitute for $5x$, in the ſecond equation, its value thus found, and we have equat. 5. $\frac{480+10y}{3}+4y=180$. This equation contains only one unknown quantity, and being ſolved by the common rules gives $y=30$, as before.

Sometimes one method, ſometimes another is to be choſen, according to the nature of the problem.

155. PROBLEM 18. Divide 20 into two ſuch parts, ſo that a third of the one part, added to a fifth of the other may make 6.

For the two parts put x and y, and we have

Fundamental equation $\begin{cases} x+y=20 \\ \frac{x}{3}+\frac{y}{5}=6 \end{cases}$

Secondary equations $\begin{cases} \text{1ſt.} & x+y=20 \\ \text{2d.} & 5x+3y=90. \end{cases}$

From equat. 1. we have equat. 3. $y=20-x$; for y, or rather for $3y$, in equat. 2. ſubſtitute its value, according to method third, and we have

Equat. 4. $5x+60-3x=90=60+2x$,

Whence $x=15$, and by equat. 3. $y=5$.

156. PROBLEM 19. Says A to B, give me five ſhillings of your money, and I ſhall have twice as much as you will have left. Says B to A, rather give me five ſhillings of your money, and I ſhall have 3 times as much as you will have left. What had each?

For A's money put x, for B's money put y.

Fun-

Fundamental equat. $\left\{\begin{array}{l} x+5=2\times\overline{y-5} \\ y+5=3\times x-5 \end{array}\right\}$ whence

Equat. 1. $x+5=2y-10$

Equat. 2. $3x-15=y+5$.

Double this (by method first) and we have

Equat. 3. $6x-30=2y+10$.

Subtract equat. 1. from equat. 3. and $5x-35=*+20$, whence $x=11$. This substituted for x, in equat. 1. gives $11+5=2y-10$, whence $y=13$.

After the first exchange, *A* had 16 shillings, and *B* 8 shillings, but $16=2\times 8$. After the second exchange, *A* had 6 shillings, and *B* 18 shillings, but $3\times 6=18$, which is the proof.

157. PROBLEM 20. From the Ladies' Diary 1708.

When first the marriage knot was tied
 Betwixt my wife and me,
My age to her's we found agreed,
 As nine doth unto three:
But after ten and half ten years,
 We man and wife had been,
Her age came up as near to mine,
 As eight is to sixteen.
Now tell me, if you can, I pray,
What was our age o'th' marriage day?

For the man's age put x, for the woman's y.

Then $\left\{\begin{array}{l} x : y :: 9 : 3, \text{ or as } 3 : 1 \\ x+15 : y+15 :: 16 : 8, \text{ or as } 2 : 1. \end{array}\right.$

Whence equat. 1. $x=3y$

Equat. 2. $x+15=2y+30$.

For x in the second equation, substitute its value found in the first equation, and we have $3y+15=2y+30$, whence $y=15$, and $x=45$.

158. It is manifest, that the difference of their ages must always be the same, namely 30 years, but though the difference between two numbers is always the same, yet their proportion to each other alters, as might be learnt from problem 12. par. 130. In the present case, 45 is three times 15; add 15 to each number, and the former number becomes 60, the latter

latter 30; but 60 is only the double of 30. Thus we ſee the difference between two numbers may be the ſame, and yet their proportion alter; and contrariwiſe, the proportion between two numbers may be the ſame, yet their difference alter. Thus the proportion between 2 and 3 (or rather of 2 to 3) is the ſame as that of 8 to 12, but the difference of the two former numbers is only 1, that of the latter two numbers is 4.

159. Problem 21. A jockey has two horſes *A* and *B*. He has alſo two ſaddles, one valued at 16 pounds, the other at 4: now if he ſets the better ſaddle upon *A*, and the worſe upon *B*, then *A* will be worth twice as much as *B*; but if he ſets the better ſaddle upon *B*, and the worſe upon *A*, then *B* will be worth three times as much as *A*. I demand the values of the horſes.

For the value of *A* put x, for *B* put y.

Fundam. equat. $\begin{cases} x+16=2\times\overline{y+4} \text{ or } x+16=2y+8=\text{eq. 1.} \\ y+16=3\times\overline{x+4} \text{ or } 3x+12=y+16=\text{eq. 2.} \end{cases}$

From equat. 2. we have $3x-4=y$ and $6x-8=2y$. Subſtitute this for $2y$ in equat. 1. and $x+16=6x+8-8=6x$, whence $5x=16$ and $x=3\frac{1}{5}=\overset{\pounds.}{3}.\overset{s.}{4}$, and $y=3x-4=5\frac{3}{5}=\overset{\pounds.}{5}.\overset{s.}{12}$, as will appear on trial.

For in the firſt caſe, *A* is worth $\overset{\pounds.}{3}.\overset{s.}{4}+\overset{\pounds.}{16}.\overset{s.}{0}=\overset{\pounds.}{19}.\overset{s.}{4}$, and *B* is worth $\overset{\pounds.}{5}.\overset{s.}{12}+\overset{\pounds.}{4}.\overset{s.}{0}=\overset{\pounds.}{9}.\overset{s.}{12}$; but $\overset{\pounds.}{19}.\overset{s.}{4}=$ twice $\overset{\pounds.}{9}.\overset{s.}{12}$. In the ſecond caſe, *A* is worth $\overset{\pounds.}{3}.\overset{s.}{4}+\overset{\pounds.}{4}.\overset{s.}{0}=\overset{\pounds.}{7}.\overset{s.}{4}$, and *B* is worth $\overset{\pounds.}{5}.\overset{s.}{12}+\overset{\pounds.}{16}.\overset{s.}{0}=\overset{\pounds.}{21}.\overset{s.}{12}=$ three times $\overset{\pounds.}{7}.\overset{s.}{4}$.

160. Problem 22. A certain company at a tavern, when they came to pay their reckoning, found that had there been four more in company they might have paid a ſhilling a piece leſs than they did; and

that if there had been three fewer in company, they muſt have paid a ſhilling a piece more than they did. I demand the number of perſons, what each paid, and what was the reckoning.

Here it may be obſerved, that there are but two unknown quantities independent of each other: for of the three mentioned, *viz.* the number of perſons, their *quota*, and the reckoning; if any two be known, the third may be inferred from them. Put then x for the number of perſons; and becauſe their *quota* muſt be leſs than the whole reckoning, let us make the *quota* the other unknown quantity, and put for it y, that is, let y be the number of ſhillings actually paid by each, and xy will repreſent the whole reckoning.

Then on the firſt ſuppoſition, that the number of perſons had been $x+4$, each muſt have paid $\frac{xy}{x+4}$, which by the condition of the problem is one ſhilling leſs than they actually paid, or is equal to $y-1$. Again, on the ſecond ſuppoſition, that the number of perſons had been $x-3$, each muſt have paid $\frac{xy}{x-3}$, which by the problem is one ſhilling more than they actually paid, or is $=y+1$; therefore the fundamental equations ſtand thus

$$\frac{xy}{x+4}=y-1$$

$$\frac{xy}{x-3}=y+1,$$

and the ſecondary equations are $0=4y-x-4$ Eq. 1.

$0=x-3y-3$ Eq. 2.

From equation firſt we have $x=4y-4$ Eq. 3.

From equation ſecond $x=3y+3$ Eq. 4.

Hence $4y-4=3y+3$, and $y=7$ ſhillings; and from equation 3 or 4, we have $x=24$ perſons, whence the reckoning is 168 ſhillings. Now had there been 4 more, that is 28 perſons in all, they might have paid only 6 ſhillings a piece; but had there been 3

fewer,

fewer, that is 21 persons, they must have paid 8 shillings a piece.

161. By methods like these may problems, having three or more unknown quantities, be solved. But this (and indeed all the more abstruse parts of algebra) is of so little use in the study of natural philosophy, that we shall pass it over. Besides the solution of such problems, often depends on some artful substitution, peculiar to that problem only (and perhaps found out by chance) and therefore of no use in other cases. We shall give one instance as a specimen of this sort of ingenuity.

162. PROBLEM 23. To find four numbers with the following properties. The sum of the three first is 13. The sum of the two first and last is 17. The sum of the first and two last is 18, the sum of the three last 21.

Put a, b, c, d, for the four numbers respectively, and the fundamental equations will stand thus;

$$a+b+c=13$$
$$a+b+d=17$$
$$a+c+d=18$$
$$b+c+d=21.$$

Substitute S for the sum of all the four numbers; that is, put S for $a+b+c+d$, and the fundamental equations will be transformed into the following ones.

Fundamental equations $S-d=13$

$$S-c=17$$
$$S-b=18$$
$$S-a=21.$$

Add all these equations together, and we have

$$4S-a-b-c-d=69, \text{ that is}$$
$$4S-\overline{a+b+c+d}=69, \text{ or}$$
$$4S-S=69, \text{ that is } 3S=69, \text{and } S=23,$$

For S put its value in the four transformed equations, and we shall have, first $23-a=21$, and $a=2$; then $23-b=18$, and $b=5$; again, $23-c=17$, and $c=6$; lastly $23-d=13$, and $d=10$: therefore 2, 5, 6, and 10, are the four numbers sought.

 163. There

163. There is a caſe in ſimple equations, of which we have taken no notice. The deſcription of this caſe, and the rules for proceeding in it, makes a *fifth proceſs* to be now added to the other four formerly laid down, par. 63—66.

PROCESS 5. If the whole equation can be divided by the unknown quantity, let ſuch a diviſion be made, and the equation will be reduced to a more ſimple one. Of this we ſhall give one example only.

164. PROBLEM 24. What two numbers are thoſe, whereof the leſs is to the greater as 2 to 5, and the product of whoſe multiplication is ten times the ſum of the numbers?

For the leſs put x, for the greater y;

And $2:5::x:y$, whence $2y=5x$: Eq. 1.

Again $xy=10\times\overline{x+y}=10x+10y$. Eq. 2.

From the firſt equation we have $y=\frac{5x}{2}$: ſubſtitute this for y in the ſecond equation, and $\frac{5xx}{2}=10x+\frac{50x}{2}=\frac{20x+50x}{2}$: hence $5xx=20x+50x$. Here divide the whole equation by x, the unknown quantity, and it will be reduced to $5x=20+50=70$, and $x=14$: whence $2:5::14:35=y$. Now the ſum of 14 and 35 is 49, and $14\times35=490$, or 10 times 49.

OF QUADRATIC EQUATIONS.

165. Equations containing only one unknown quantity, being firſt reduced as far as is poſſible by the fifth proceſs, are then claſſed according to the dimenſions of that unknown quantity. If the unknown quantity is found ſimply, or in the firſt power, it is called a ſimple equation. Of this ſort are all thoſe we have before met with. The fundamental equations of every problem, when reduced, finally give the value of x ſimply, in the firſt power. But if after due

due reduction, the square of the unknown quantity, or xx, is found in the final equation, that equation is called a quadratic. If the cube xxx, is found, it is called a cubic equation, and so on.

166. Quadratic equations are divided into two sorts: First, *Pure quadratics*, where the square of the unknown quantity is found alone. Secondly, *Affected quadratics*, where the square of the unknown quantity is found with, or affected with, its side or root.

THE SOLUTION OF PURE QUADRATICS.

167. All the former processes are to be applied, but one more is to be added.

PROCESS 6. If in the final equation, the square of the unknown quantity appears to be equal to some known quantity, extract the square root of both sides of the equation, and we shall have the simple value of the unknown quantity on one side, equal to the square root of the known quantity on the other.

Of the use of this rule we shall give two examples.

168. PROBLEM 25. Given the product of the multiplication of two numbers 144, and the quotient of the greater divided by the less $=9$. To find the two numbers.

Let the greater number be x, and the less y. Then we have equat. 1. $xy=144$.

and equation 2. $\frac{x}{y}=9$.

From the 2d equat. we have equat. 3. $x=9y$. Substitute for x in the first equation, its value found in the third equation, and we have $9yy=144$. Divide both sides by 9 (the co-efficient of yy) and we have $yy=16$. Extract the square root on both sides, and we have $y=4$, and from the third step $x=9\times4=36$, as is evident; for $36\times4=144$, and $\frac{36}{4}=9$.

169. PROBLEM 26. What two numbers are those,

 the

the product of whose multiplication is 108, and whose sum is twice their difference.

Let the greater number be x, the less y, and the fundamental equations are $xy = 108$.

and $x + y = 2 \times \overline{x - y} = 2x - 2y$: from whence
$2y + y = 2x - x$, that is $3y = x$.

For x, in the first fundamental equation, substitute its value now found, and we have $3yy = 108$, whence $yy = 36$, and $y = 6$. From the third step, $x = 3 \times 6 = 18$. For $18 \times 6 = 108$, and $18 + 6 = 24$, also $18 - 6 = 12$, but 24 their sum is twice 12 their difference.

Saunderson, art. 40. solves this problem by one letter only; but the solution by one letter only is far more intricate and difficult, than this by two letters, as will appear on comparing them.

THE SOLUTION OF ADFECTED QUADRATIC EQUATIONS.

Preparatory Observations.

170. DEFINITION 1. A *Binomial* is a compound quantity consisting of two members, connected by the sign + or −, thus $a + b$ is a binomial, being the sum of a and b; also $a - b$ is a binomial, being the difference of a and b.

171. DEFINITION 2. A *Trinomial* is a compound quantity consisting of three members, connected by the signs + or −, as $a + b - c$, or $aa + 2ab + bb$.

172. OBSERVATION 1. The square of a binomial is a trinomial; thus the square of $a + b$ is $aa + 2ab + bb$, and the square of $a - b$ is $aa - 2ab + bb$. And contrariwise, the square root of either of these trinomials is a binomial, for the square root of $aa + 2ab + bb$ is $a + b$, and the square root of $aa - 2ab + bb$ is $a - b$.

173. OBSERVATION 2. Of the three terms, that compose the square of a binomial, the first term is affir-

affirmative and alſo a pure ſquare, being the ſquare of the leading quantity. The third term is alſo affirmative and a pure ſquare, being the ſquare of the other, or the following quantity. The middle term is double the product of the two quantities that compoſe the binomial, and has for its ſign either + or −, according as the two members of the binomial were connected by the ſign + or −.

174. OBSERVATION 3. Hence the third term is always the ſquare of half the co-efficient of the leading quantity in the ſecond term. Thus if the firſt term of the three be aa, and the ſecond either $+2ab$ or $-2ab$; then the co-efficient of a in $2ab$ is $2b$, half this co-efficient is b, and the ſquare of that half co-efficient is bb, which is the third term of the trinomial.

175. OBSERVATION 4. Therefore if the two firſt of the three terms, which conſtitute the ſquare of a binomial, be given, the third term (which is wanting) may be found by adding to them the ſquare of half the co-efficient of the leading quantity in the ſecond term. This, in ſolving quadratic equations, is called *compleating the ſquare*.

176. OBSERVATION 5. The ſquare of the binomial being thus compleated; its root is the binomial itſelf, conſiſting of the ſquare root of the firſt term and the ſquare root of the laſt term, connected with the ſign + or −, according as the middle term is affirmative or negative.

Method of reſolving adfected quadratic Equations.

177. All the former proceſſes and rules are to be obſerved till you come to the final equation, in which the unknown quantity (and its powers) are on one ſide of the equation, and known quantities only on the other; and then we have five additional proceſſes peculiar to the ſolution of adfected quadratic equations.

178. PROCESS 1. Make the ſquare of the unknown quantity affirmative, if it be not already ſo. This is done by changing the ſign of every quantity on both ſides the equation. Alſo, make that ſquare the firſt or leading term in the equation.

179. PROCESS 2. Divide the whole equation by the co-efficient of the leading term (or ſquare of the unknown quantity) that it may be a pure ſquare, if it be not ſo already.

180. PROCESS 3. Compleat the ſquare according to the directions in Obſervation 4.

181. PROCESS 4. Extract the ſquare root on both ſides of the equation, according to Obſervation 5.

182. PROCESS 5. Tranſpoſe the known parts of the ſquare root, to that ſide of the equation which is wholly known.

Remarks on the fourth Proceſs.

183. Every affirmative quantity has two ſquare roots, one affirmative and one negative. Thus -6 is the ſquare root of 36, as well as $+6$; for $-6 \times -6 = 36$, as well as $+6 \times +6$. And in extracting the ſquare root of the known ſide of the equation, both the affirmative and negative roots muſt be taken into the account, and the ſquare root marked $\pm$; ſo that every quadratic equation will have two anſwers.

184. A negative quantity has no ſquare root, becauſe $-$ into $-$ gives $+$, as well as $+$ into $+$: ſo that when the known ſide of the equation is a negative quantity, both roots will be impoſſible, and the problem incapable of an anſwer. This will fully appear in the following examples.

185. PROBLEM 27. Divide the number 20 into two ſuch parts, that the product of their multiplication may be 64.

For one part put x, for the other y, and the fundamental equations are $x+y=20$

$$xy=64.$$

From

From the firſt of theſe we have $x=20-y$. For x in the ſecond fundamental equation, ſubſtitute its value ſo found, and we have $xy=20y-yy=64$.

Make the ſquare affirmative and $yy-20y=-64$.

Compleat the ſq. $yy-20y+100=-64+100=36$.

Extract the ſquare root; $y-10=\pm 6$.

Tranſpoſe the known part; $y=+10\pm 6=16$ or 4.

Hence from the third ſtep $x=4$ or 16.

186. Here we may obſerve, that whether we ſuppoſe x to repreſent the greater quantity and y the leſs or contrariwiſe, ſtill the fundamental equations will be exactly the ſame. Therefore to be conſiſtent with truth, the ſolution of the problem ought not to determine x to be the greater quantity rather than the leſs. And accordingly it finds two values of x, and of courſe two correſpondent values of y, according as you are pleaſed to ſuppoſe x to be the greater or leſs of the two numbers.

187. *Caſe* 2. All things remaining as before, ſuppoſe the product of their multiplication to be 96, what are the numbers in that caſe?

Following the ſteps of the foregoing ſolution, we ſhall find at laſt that $y=10\pm 2$, ſo that the two roots of the equation are 12 and 8.

188. *Caſe* 3. Again; Let the product of their multiplication be 99. In this caſe the two roots will be 11 and 9, ſo that they come nearer to an equality.

189. *Caſe* 4. Let now the product of the multiplication be 100. In this caſe $y=10\pm 0$, ſo that the two roots become equal, each being 10.

190. *Caſe* 5. Laſtly, let the product of their multiplication be 104.

Here we ſhall come to the following ſtep, that $yy-20y+100=+100-104=-4$; but -4 has no ſquare root, therefore we can proceed no further, nor obtain any anſwer to this caſe of our problem.

191. To explain this we muſt obſerve, that if we divide 20 into two parts, and make the former part 1, and

and the latter 19, the product of their multiplication will be 19. Increaſe now the former part, and decreaſe the latter by unity, and their product will be 36. Proceeding in this manner, we ſhall find the products continually increaſe, till the two parts are equal, each being 10, and their product 100. After which the products will decreaſe in the ſame manner they before increaſed. Thus 11×9 is the ſame as 9×11. So that the product will be the greateſt *poſſible* when both parts are equal. When therefore it was propoſed that their product ſhould be 104, it was propoſed that their product ſhould be greater, than the greateſt poſſible, which is abſurd. In other words, the latter condition is ſo propoſed as to be inconſiſtent with the former. For if the ſum of the two numbers, be no more than 20, the product of their multiplication can never riſe to more than 100. And this inconſiſtency is pointed out by the impoſſibility of extracting the root of a negative quantity.

192. The algebraic rules would be incompatible with truth, if they brought out a poſſible anſwer to a problem manifeſtly impoſſible. Theſe rules, if ſtrictly followed, always bring out the form of an anſwer; but then in this caſe it will be the *form* only, and that form attended with ſuch circumſtances as always mark the repugnancy of the fundamental equations.

193. Here it may be proper to remark the difference between negative and impoſſible anſwers.

A negative anſwer implies, that the problem is impoſſible in the ſtrict literal ſenſe of adding or ſubtracting, but not in the more extenſive ſenſe of algebraic addition and ſubtraction. And in this caſe there is always another problem of the ſame kind, in which the negative anſwer ſhall become affirmative, and poſſible in the ſtricteſt ſenſe. Such a problem will be found by changing the word *add* in the former problem into *ſubtract*, and contrariwiſe, changing *ſubtract* into *add*; of this we ſhall give an inſtance or two.

4 When

When impoſſible anſwers come out, then no ſuch change in the words of the problem can make it poſſible.

194. It is neceſſary here to introduce another character, put for the ſquare root of any quantity, to wit, $\surd$, thus $\sqrt{4}$ is the mark for the ſquare root of 4, therefore $\sqrt{4}=2$, likewiſe $\sqrt{aa+2ab+bb}=a+b$.

195. Some numbers have exact ſquare roots, others have not. Thus, 16 has an exact ſquare root, to wit 4. The number 25 alſo has for its ſquare root 5. But 20, which lies between them, muſt of courſe have a ſquare root greater than 4 and leſs than 5, that is 4 with ſome (decimal) fraction; but that fraction cannot be aſſigned or found exactly. The root therefore is ſignified by the forementioned character, thus $\sqrt{20}$. Such quantities are called Surds.

196. In like manner, when quantities have no ſquare root at all, yet this character is retained to ſignify an imaginary ſquare root; thus $\sqrt{-4}$ is retained to ſignify the imaginary ſquare root of -4. For -4 has no real ſquare root. Here it will be aſked, of what uſe is it to have a character for that which has no exiſtence? An example will teach us the uſe.

197. In the laſt problem we came to this concluſion, that $yy-20y+100=+100-104=-4$. Now although the ſquare root of -4 is an impoſſible quantity, yet we may retain the character for it, and ſo go on with our work. Extract the ſquare root of both ſides of this equation, and we have

$$y-10=\pm\sqrt{-4}; \text{ whence}$$

$$y=10\pm\sqrt{-4}, \text{ or } 10+\sqrt{-4} \text{ and } 10-\sqrt{-4}.$$

198. By retaining this character, we learn that both the values of y (not one only) are impoſſible, one of them being the ſum of the number 10, and an impoſſible number, the other the difference of ten and that impoſſible number, therefore it may be uſeful to retain this character, though it repreſents an impoſſibility.

199. Pro-

199. PROBLEM 28. Given the sum of two numbers 20, and the sum of their squares 272, to find the numbers.

For one number put x, for the other y, and the fundamental equations are $x+y=20$

$$xx+yy=272.$$

From the former of these we have $x=20-y$, and $xx=400-40y+yy$. Substitute this for xx in the second fundamental equation, and we have

$$2yy-40y+400=272;$$

whence $2yy-40y=272-400=-128$.

By process 2. $yy-20y=-64$.

By process 3. $yy-20y+100=100-64=36$.

By process 4. $y-10=\pm 6$.

By process 5. $y=10\pm 6=16$ or 4; therefore $x=4$ or 16, as before: see par. 186.

200. PROBLEM 29. A certain company had a reckoning of 7£. 4s. to pay; upon which two of the company sneaking off, obliged the rest to pay one shilling a piece more than they should have done. What was the number of persons?

First we must observe, that the reckoning reduced to shillings makes 144. Let the number of persons at first be x, and their quota will be $\frac{144}{x}$. But after two were gone away, their number was reduced to $x-2$, and then their quota will be $\frac{144}{x-2}$: now by the question this is one shilling more than what they should have paid, that is $\frac{144}{x-2}=\frac{144}{x}+1$. This equation, cleared of fractions, and the unknown quantities brought to one side, gives $xx-2x=288$;

whence by process 3. $xx-2x+1=288+1=289$.

by process 4. $x-1=\pm 17$.

by process 5. $x=1\pm 17=+18$ or -16.

201. That the number of persons was 18, is evident, for then they pay 8 shillings a piece; but when their

their number was reduced to 16, they muſt pay 9 ſhillings a piece, or one ſhilling more than they ſhould have done. About the affirmative anſwer, there is no difficulty; but what are we to think of the negative one? What is the meaning of -16 perſons? That -16 will anſwer the algebraic conditions of the problem is plain; for if -16 perſons pay -9 ſhillings a piece, it will amount to $+144$ ſhillings, for $-16 \times -9 = +144$. Again, ſubtract $+2$ from -16, and the (algebraic) remainder is -18, and if -18 perſons pay -8 ſhillings a piece, it will alſo amount to $+144$ ſhillings: moreover, ſubtract, as before, what they paid in the former caſe, from what they paid in the latter, that is, ſubtract -9 from -8, and the (algebraic) difference is $+1$, according to the conditions of the problem. But though theſe negative numbers anſwer the algebraic conditions, ſtill it will be aſked, what is the meaning of all this? The anſwer to this queſtion has been already given in par. 193. According to what we there laid down, there is another problem of the like ſort anſwered at the ſame time with this. And in this problem, the perſons inſtead of paying money, muſt receive money (for if money paid be called affirmative money, then negative money muſt ſtand for money received). Inſtead of two going out, we muſt ſuppoſe two to come in, &c. A problem of this ſort may be propoſed in the following form.

202. Problem. A certain member of parliament ordered his ſteward to diſtribute 7*l*. 4*s*. among the poor voters. When they were aſſembled together, and the ſteward ready to diſtribute the money, there comes in unexpectedly two more claimants; by which means, thoſe at firſt aſſembled received one ſhilling a piece leſs than they otherwiſe would have done. What was the number at firſt;

Here the number of ſhillings to be diſtributed is 144, anſwering to what we had before. Let the number of perſons at firſt be x, and the quota which they would

would have received will be $\frac{144}{x}$: but when two more were come in, their number will be increaſed to $x+2$, and their quota will be $\frac{144}{x+2}$. Now this is one ſhilling leſs than what they would have received, that is $\frac{144}{x+2} = \frac{144}{x} - 1$. This equation reduced as before gives $xx+2x=288$;

whence by proceſs 3. $xx+2x+1=288+1=289$.

by proceſs 4. $x+1=\pm 17$.

by proceſs 5. $x=-1\pm 17=+16$ or -18.

So the number of perſons was 16. What was the negative anſwer in the former problem, is the affirmative anſwer in this; contrariwiſe, the negative anſwer here, is the affirmative one in the former problem. We can hardly ſay with Saunderſon, that negative roots have no place in this ſort of problems. (Saunderſon's Algebra, art. 117.) We ſhall give another inſtance of the like ſort.

203. Problem 30. What number is that, which added to its ſquare, makes 72?

Put for the number x, and its ſquare will be xx, whence this equation $xx+x=72$

by proceſs 3. $xx+x+\frac{1}{4}=72+\frac{1}{4}=\frac{289}{4}$

by proceſs 4. $x+\frac{1}{2}=\pm\frac{17}{2}$

by proceſs 5. $x=-\frac{1}{2}\pm\frac{17}{2}=+\frac{16}{2}$ or $\frac{-18}{2}=+8$ or -9.

And either of theſe anſwers ſatisfy the conditions of the problem, not indeed literally, but algebraically: for $8\times 8=64$, to this add 8, and it makes 72. So again, $-9\times -9=+81$, to this add (algebraically) -9, and it makes 72. But the algebraic addition of -9, is the ſame as the arithmetical ſubtraction of $+9$. If then we take the words of the problem literally, the negative anſwer has no place; but then there is an-

another problem, in which it does literally take place; viz. What number is that which being ſubtracted from its ſquare leaves 72? Or the problem might have been propoſed in this way. What number is that, which being *incorporated* with its ſquare, makes 72? and then it will have two anſwers: 8 is the number, if the incorporation is to be made by addition; and 9 is the number, if the incorporation is to be made by ſubtraction.

204. *Lemma.* In the ſeries of odd numbers (beginning with unity) as 1, 3, 5, 7, 9, &c. the ſum of any number of terms, n, will be nn, or the ſquare of the number of terms given.

Examples. The ſum of four terms, to wit, 1, 3, 5 and 7, is 4×4 or 16. The ſum of ſeven terms, or 1, 3, 5, 7, 9, 11, 13, is 7×7 or 49.

In general, the continual addition of the odd numbers produces the ſeries of ſquare numbers.

205. PROBLEM 31. A traveller A ſets out from a certain place and travels one mile the firſt day, three miles the ſecond day, five the third day, and ſo on according to the ſeries of the odd numbers. Eight days after another, B, ſets out, and travels the ſame road at the rate of 36 miles every day. I demand how long and how far A muſt travel before he is overtaken by B.

Put x for the number of days A travels before he is overtaken by B. Then to find how far A travels in that time, I obſerve the number of miles he travels in x days, is the ſum of a ſeries of odd numbers, beginning with unity; the number of terms in that ſeries being x, or the number of days: but the ſum of x terms is xx by the lemma, therefore the miles travelled by A is xx. Again, B travels $x-8$ days, and conſequently at 36 miles a day, B will have travelled $\overline{x-8} \times 36$ or $36x-288$; whence this fundamental

equation $xx = 36x - 288$

and $xx - 36x = -288$;

by proceſs 3. $xx - 36x + 324 = 324 - 288 = 36$

by proceſs 4. $x - 18 = \pm 6$

by proceſs 5. $x = 18 \pm 6 = 12$ or 24.

Here

Here then are two affirmative anſwers, and either anſwer the fundamental equation.

For when A has travelled 12 days, he will have gone 144 miles by the lemma. At the ſame time B will have travelled 4 days at the rate of 36 miles a day, therefore he will alſo have gone 144 miles. Again, when A has travelled 24 days, he will have gone 24×24 or 576 miles: at the ſame time B will have travelled 16 days at the rate of 36 miles a day; therefore he will alſo have gone 576 miles.

206. The equation in this problem is derived from a condition more general, than the words of it imply, namely, from the equality of the diſtance of the two travellers from the place from whence they ſet out, as was obſerved in a like caſe before; ſee prob. 16. When B firſt ſets out he is the ſwifter traveller, going 36 miles in the firſt day of his journey; whereas this being the 9th day with A he goes only 17 miles in that day; and therefore B gaining ground upon A overtakes him, the 12th day of A's journey, on which day A travels 23 miles; B therefore not only overtakes A, but alſo paſſes him. But on the 19th day A becomes the ſwifter traveller, going that day 37 miles, and increaſing his rate every day 2 miles, he muſt of neceſſity overtake B again. Hence ariſes the double anſwer, both of which ſatisfy the condition from which the equation was drawn, though the former only anſwers the letter of the problem, that B is to *overtake* A, and not A overtake B.

207. It ſhould alſo be obſerved, that if one anſwer is poſſible, then both are poſſible, and that there can be no more than two anſwers. For if ever B paſſes A, then A muſt of neceſſity paſs B. For how ſwift ſoever be the rate of B's travelling, as the rate of A increaſes every day by two miles, it muſt ſome time or other, not only equal the rate of B, but ſurpaſs it more and more; the rate of B being always the ſame. But when A has thus paſſed B, his rate being more than that of B and increaſing daily, A muſt of neceſſity leave B more

more and more behind, ſo that they will never more be together again.

208. *Caſe* 2. All things as before, let *B* ſet out 9 days after *A*, when will *B* overtake *A* in that caſe?

Following the ſteps of the problem, as before, we ſhall find $x=18\pm0$, that is, the two values of x are equal, and the two anſwers, or places where they are together, coaleſce into one. In this caſe *B* ſtays ſo long before he ſets out, that he cannot overtake *A* till the end of the 18th day, after which *A* being the ſwifter traveller, he will leave *B* more and more behind for ever.

209. *Caſe* 3. All things as before, let *B* ſet out ten days after *A*. When will *B* overtake *A* in that caſe?

Following the ſame ſteps as before, we ſhall find $x=18\pm\sqrt{-36}$; but $\sqrt{-36}$ is an impoſſible quantity, and therefore both roots impoſſible. In this caſe, *B* ſtays ſo long behind, that he cannot come up with *A* by the 18th day at night. But after the 18th day, *A* travelling faſter than *B*, and increaſing his rate every day (while *B* contiuues at the ſame rate) it becomes impoſſible for *B* to overtake *A* at all; much leſs to paſs by *A*. And thus both anſwers will be impoſſible, which is pointed out, becauſe both roots are impoſſible *.

* Beſides the method of ſolving adfected quadratic equations by completing the ſquare, as before taught, there is another, viz. by taking away the ſecond term ($\pm px$) in the equation $xx\pm px=q$, and ſo reducing it to a pure quadratic. This is done by ſubſtituting $y\mp\frac{p}{2}$ for x in that equation; and is ſimilar to what is done in cubic equations. But we thought the common method ſufficient for a learner.

OF GENERAL PROBLEMS.

210. Hitherto we have admitted both numbers and letters into problems. Of courſe the ſolution of ſuch is particular; ſuited only to the particular numbers ſtated or laid down in the conditions. But it is of the greateſt uſe to reaſon more generally, and to give a ſolution, from which a general Rule or Theorem may be drawn, for ſolving all particular caſes of problems of the like kind. Such a general inveſtigation of the unknown quantity, is called the ANALYSIS, or *Analytical inveſtigation* of the problem. The aſſuming the value of the unknown quantity (in known terms), ſuch as the problem finally determines that value to be; and ſhowing that the quantities ſo aſſumed have the properties deſcribed in the problem, is called the SYNTHETICAL DEMONSTRATION. The tranſlation of the value of the unknown quantity (in known terms) out of the algebraic into common language, is deducing a THEOREM, or forming a CANON, for all caſes of the like kind, according to the form of words in which the tranſlation is made. If the equality between the unknown quantity and its value (in known terms) be ſimply declared; it is a Theorem. If the arithmetical operations for computing the value of the unknown quantity (as algebraically expreſſed) be laid down in words at length, this is called a Canon or Rule for computing the value of the quantity ſought in all queſtions of the like kind. Of all this we ſhall give one compleat example; though all problems muſt be for the future, not only analytically inveſtigated, but ſynthetically demonſtrated.

211. PROBLEM 32. What two numbers are thoſe whoſe ſum is s and difference d.

Put x for the greater number and y for the leſs, and the two fundamental equations will be $x+y=s$ and

and $x-y=d$. From the latter of theſe we have $x=d+y$. Subſtitute this for x in the former fundamental equation, and we have $2y+d=s$, and $y=\frac{s-d}{2}$. Whence $x=d+\frac{s-d}{2}=\frac{2d}{2}+\frac{s-d}{2}=\frac{s+d}{2}$: ſo the greater number x, is $\frac{s+d}{2}$, and the leſs y, is $\frac{s-d}{2}$.

212. *Synthetical Demonſtration.* $\frac{s+d}{2}+\frac{s-d}{2}=\frac{2s}{2}=s$. Again $\frac{s+d}{2}-\frac{s-d}{2}=\frac{2d}{2}=d$.

213. Hence THEOREM I. *The difference between any two numbers added to their ſum is equal to twice the greater.*

214. THEOREM 2. *The difference between any two numbers, ſubtracted from their ſum, is equal to twice the leſs.*

215. PROBLEM: It is required, having given the ſum and difference of any two numbers, to find the numbers themſelves.

CANON. *Add the difference to the ſum and half the aggregate will be the greater number.* Again, *Subtract the difference from the ſum and half the remainder will be the leſs number.*

216. *Example.* Let their ſum be 19 and their difference 10. What are their numbers? Here $s+d=29$ and $\frac{s+d}{2}=\frac{29}{2}=14\frac{1}{2}=$ the greater. Again $s-d=9$ and $\frac{s-d}{2}=\frac{9}{2}=4\frac{1}{2}=$ the leſs: for $14\frac{1}{2}+4\frac{1}{2}=19$, and $14\frac{1}{2}-4\frac{1}{2}=10$ as required.

217. *Remark.* We are at liberty to change the form of the general algebraic expreſſions, which determine x and y, provided we do not change their value. Thus we may ſeparate the two members which

compose the numerators of the fractional values of x and y, and they will stand in this form: $x=\frac{s}{2}+\frac{d}{2}$ and $y=\frac{s}{2}-\frac{d}{2}$. This will afford great variety of expression in drawing out a theorem or canon. It may now stand thus: *To the semi-sum of the two numbers, add their semi-difference, and it makes the greater; and from the semi-sum subtract their semi-difference, and it leaves the less number.*

218. PROBLEM 33. Given the difference of two numbers d, and the difference of their squares b; to find the numbers.

Let the greater number be x, and the less y, as before, and the fundamental equations are
$$x-y=d$$
$$xx-yy=b$$
from the first equation $x=d+y$, and $xx=dd+2dy+yy$. Substitute this for xx in the second equation, and we have $dd+2dy=b$, and this solved gives $y=\frac{b-dd}{2d}$. Substitute this for y in the third step, and we have $x=d+\frac{b-dd}{2d}=\frac{2dd}{2d}+\frac{b-dd}{2d}=\frac{b+dd}{2d}$.

219. SYNTHESIS. $\frac{b+dd}{2d}-\frac{b-dd}{2d}=\frac{2dd}{2d}=d$.

Again $\overline{\frac{b+dd}{2d}}\Big|^2=\frac{bb+2bdd+d^4}{4dd}$,

and $\overline{\frac{b-dd}{2d}}\Big|^2=\frac{bb-2bdd+d^4}{4dd}$.

Subtract the latter from the former, and we have the difference of the squares of these quantities $=\frac{4bdd}{4dd}=b$.

Instead of $\frac{b+dd}{2d}$ and $\frac{b-dd}{2d}$, we may put $\frac{b}{2d}+\frac{1}{2}d$ and

and $\frac{b}{2d} - \frac{1}{2}d$, and draw a canon from either expreſſion.

220. *Example.* Let the difference of the two numbers be 10, and the difference of their ſquares 120. See prob. 10.

Here $b = 120$, $2d = 20$; $\frac{b}{2d} = \frac{120}{20} = 6$; $\frac{1}{2}d = 5$; $6 + 5 = 11$; $6 - 5 = 1$.

221. PROBLEM 34. Let r and s be two given multiplicators. It is required to divide a given number as a, into two ſuch parts, called x and y, that r times x added to s times y may make ſome other given number as b.

Here the fundamental equations are $x + y = a$ and $rx + sy = b$. From the former we have $y = a - x$, and $sy = as - sx$; ſubſtitute this for sy in the latter fundamental equation, and we have $rx - sx + as = b$, and $rx - sx = b - as$; whence $x = \frac{b - as}{r - s}$.

222. When any quantity, as x, is found in every member on one ſide of the equation, it is a ſign that the co-efficient of that quantity x is a compound quantity; and it conſiſts of the co-efficient of x in every member ſingly; connected with their proper ſigns. The co-efficient of x in our caſe then is $r - s$; for $x \times \overline{r - s} = rx - sx$; therefore divide both ſides of the equation by $r - s$ (according to proceſs 4) and we have $x = \frac{b - as}{r - s}$.

Having found x, we have $y = a - x = a - \frac{b - as}{r - s} = \frac{ar - as}{r - s} - \frac{b - as}{r - s} = \frac{ar - b}{r - s}$.

223. If r be taken greater than s, then is the denominator of the two fractional values of x and y affirmative. And in this caſe ar is alſo greater than as.

Now if b lie between ar and as, then are the two numerators of the fractions (to wit, $b-as$ and $ar-b$) alſo affirmative, and therefore both theſe fractional values of x and y are affirmative: but if b be leſs than as or greater than ar, then will x in the former caſe, or y in the latter, be negative.

The like rule holds good if s be greater than r, and of courſe as greater than ar. For then the denominator $r-s$ is negative, and if b lie between ar and as, the two numerators will be alſo negative, and therefore (by the rule for the ſign of the quotient in diviſion) the whole fraction giving the values of x and y will be affirmative.

224. *Example* 1. Let $a=20$, $r=3$, $s=5$, $b=74$, as in problem 7. Then $r-s=-2$, $as=100$, $ar=60$, and $x=\frac{74-100}{-2}=\frac{-26}{-2}=+13$. Again, $y=\frac{60-74}{-2}=\frac{-14}{-2}=+7$.

225. *Example* 2. Let $a=20$, $r=5$, $s=3$, $b=114$. Then $r-s=2$, and $as=60$. $\frac{b-as}{r-s}=\frac{114-60}{2}=\frac{54}{2}=27$. Again $ar=100$; $\frac{ar-b}{r-s}=\frac{100-114}{2}=\frac{-14}{2}=-7$. So the two parts or numbers are $+27$ and -7, that is, the latter number is not to be added to, but ſubtracted from the former to make 20. Alſo 3 times the latter is not to be added to, but ſubtracted from 5 times the former, to make 114.

226. SYNTHESIS. $\frac{b-as}{r-s}+\frac{ar-b}{r-s}=\frac{ar-as}{r-s}=\frac{a\times\overline{r-s}}{r-s}=a$.

$r\times\frac{b-as}{r-s}=\frac{br-ars}{r-s}$, and $s\times\frac{ar-b}{r-s}=\frac{ars-bs}{r-s}$, whence $\frac{br-ars}{r-s}+\frac{ars-bs}{r-s}=\frac{br-bs}{r-s}=\frac{b\times\overline{r-s}}{r-s}=b$.

227.

227. PROBLEM 35. To divide a given number a, into two ſuch parts, that one part may be to the other as r to s. Or, as it is ſometimes expreſſed: To divide a given number in a given proportion.

Let the two parts be x and y, and we have for the firſt fundamental equation $x+y=a$. Moreover $x:y::r:s$, which analogy gives for the ſecond fundamental equation $sx=ry$. From the former of theſe we have $y=a-x$, whence by ſubſtitution $sx=ar-rx$, therefore $rx+sx=ar$, and $x=\frac{ar}{r+s}$. Again $y=a-x$ $=a-\frac{ar}{r+s}=\frac{ar+as-ar}{r+s}=\frac{as}{r+s}$. Thus we have the value of $x=\frac{ar}{r+s}$ and $y=\frac{as}{r+s}$. But as proportion is concerned, it may be proper to turn theſe equations into analogies; thus, $r+s:a::r:x$ and $r+s:a::s:y$ (ſee par. 129): that is, as the ſum of the two numbers r and s is to the whole number to be divided; ſo is r to x, the part analagous to r; and ſo is s to y, the part analogous to s.

228. SYNTHESIS. $\frac{ar}{r+s}+\frac{as}{r+s}=\frac{ar+as}{r+s}=a$.

Again $\frac{ar}{r+s}:\frac{as}{r+s}::ar:as::r:s$.

229. PROBLEM 36. To find a number x, which being ſeverally added to two given numbers a and b, will make the former ſum to the latter, as r to s.

Analogy by the problem $a+x:b+x::r:s$.

Fundamental equation $br+rx=as+sx$.

Hence $rx-sx=as-br$.

By par. 222. $x=\frac{as-br}{r-s}$.

230. SYNTHESIS. $\frac{as-br}{r-s}+a=\frac{ar-br}{r-s}$

$\frac{as-br}{r-s}+b=\frac{as-bs}{r-s}$

$\frac{ar-br}{r-s} : \frac{as-bs}{r-s} :: ar-br : as-bs :: r \times \overline{a-b} : s \times \overline{a-b}$
$:: r : s.$

231. *Example.* See problem 12. Let $a=36$, and $b=52$: and firſt, let r be to s as 3 is to 4. Here $as=144$, $br=156$, $as-br=-12$, $r-s=-1$, and $\frac{-12}{-1}=+12$, as was found before. Again; Let $r:s :: 2:3$, and $as=108$, $br=104$; $as-br=+4$, $r-s=-1$, and $\frac{+4}{-1}=-4$, as was found in the ſecond caſe. Laſtly, let $r:s::2:1$, and $as=36$, $br=104$, and $as-br=-68$, $r-s=+1$, and $\frac{-68}{+1}=-68$, as before.

232. When $as=br$, then x is nothing; but when $as=br$, then $a:b::r:s$; that is, if the propoſed numbers a and b are to each other, as r is to s, then nothing is required, either to be added to, or ſubtracted from, the propoſed numbers, to give them the deſired proportion; they have it already.

Suppoſing r greater than s, then the denominator $r-s$ is affirmative; and the numerator will be affirmative or negative, as as is greater or leſs than br; and ſo will the value of the whole fraction. Contrariwiſe if r be leſs than s.

The following example may ſhew the learner how to proceed in the ſolution of adfected quadratic equations, when they occur in general problems.

233. PROBLEM 37. Given the ſum of two numbers s, and the ſum of their cubes b; to find the numbers.

Let the numbers be x and y, and the fundamental equations are $x+y=s$, and $x^3+y^3=b$. From the firſt fundamental equation we have $y=s-x$; whence $y^3=s^3-3ssx+3sxx-x^3$: to this add x^3, and we have the ſum of their cubes $s^3-3ssx+3sxx$, which by the problem $=b$; whence $-3ssx+3sxx=b-s^3$, which is plainly an adfected quadratic equation, therefore

 by

by proceſs 1. $3sxx - 3ssx = b - s^3$

by proceſs 2. $xx - sx = \frac{b - s^3}{3s}$

by proceſs 3. $xx - sx + \frac{ss}{4} = \frac{b - s^3}{3s} + \frac{ss}{4}$

234. The ſquare being thus compleated; in this and all other caſes of general problems, ſubſtitute for for the known ſide of the equation (whatever it may be) $\frac{RR}{4}$, and the laſt ſtep will ſtand thus;

by proceſs 3. $xx - sx + \frac{ss}{4} = \frac{b - s^3}{3s} + \frac{ss}{4} = \frac{RR}{4}$

by proceſs 4. $x - \frac{s}{2} = \pm \frac{R}{2}$

by proceſs 5. $x = \frac{s \pm R}{2}$.

235. We muſt now find the value of R; or rather of RR. Now we had $\frac{b - sss}{3s} + \frac{ss}{4} = \frac{RR}{4}$.

Bring this out of fractions. Multiplying by 4 we have $\frac{4b - 4sss}{3s} + ss = RR$. Unite the fractional and integral part, and we have $\frac{4b - 4sss}{3s} + \frac{3sss}{3s} = \frac{4b - s^3}{3s} = RR$, and then the anſwer will ſtand in this form. *Make* $\frac{4b - s^3}{3s} = RR$, *and the two numbers will be* $\frac{s \pm R}{2}$. See par. 186.

236. *Example*. Let the ſum of the numbers be 12, and the ſum of their cubes 468. Here $s = 12$, $b = 468$, $4b = 1872$; $s^3 = 1728$; $3s = 36$. $\frac{4b - s^3}{3s} = \frac{1872 - 1728}{36} = \frac{144}{36} = 4 = RR$; therefore $R = 2$, and $\frac{s + R}{2} = \frac{12 + 2}{2} = 7$, and $\frac{s - R}{2} = \frac{12 - 2}{2} = 5$.

Once

Once more; let the ſum of the numbers be 3, and the ſum of their cubes 9. Here $s=3$, $3s=9$; $b=9$, $4b=36$; $sss=27$; $\frac{4b-s^3}{3s}=\frac{36-27}{9}=\frac{9}{9}=1$; that is $RR=1$, and therefore $R=1$; and $\frac{s+R}{2}=\frac{3+1}{2}=2$, and $\frac{s-R}{2}=\frac{3-1}{2}=1$.

237. SYNTHESIS. $\frac{s+R}{2}+\frac{s-R}{2}=\frac{2s}{2}=s$.

Cube of $\frac{s+R}{2}=\frac{s^3+3ssR+3sRR+R^3}{8}$.

Cube of $\frac{s-R}{2}=\frac{s^3-3ssR+3sRR-R^3}{8}$.

Their ſum $\frac{2s^3 \quad * \quad +6sRR \quad *}{8}=\frac{s^3+3sRR}{4}$.

For $3sRR$, write its value, to wit, $4b-s^3$, and we have the ſum of their cubes $=\frac{s^3+4b-s^3}{4}=\frac{4b}{4}=b$.

Here we may obſerve, that if s^3 is greater than $4b$, RR is negative, and the problem impoſſible, the two conditions being inconſiſtent. It is proper thus to ſettle the *limits* of every problem producing a quadratic equation.

238. We ſhall now give a ſet of problems related to each other, and intended not only as examples of the foregoing rules; but alſo as examples of the application of algebra to the ſolution of geometrical problems. At preſent we can only enter upon the algebraic part.

Let x and y be any two numbers, whereof x is the greater and y the leſs.

Let their ſum $x+y$ be s
the difference $x-y=d$
their product $xy=p$
their quotient $\frac{x}{y}=q$;

the

the ſum of their ſquares $xx+yy=a$.

the difference of their ſquares $xx-yy=b$.

Then any two of theſe ſix being given, the numbers may be found; or any of the other quantities without finding the numbers themſelves. Thus, if s and d be given, and it is required to find the product of the multiplication of thoſe numbers (whoſe ſum is s and difference d) we ſhall find it to be $\frac{ss-dd}{4}$. For by prob. 32, $x=\frac{s+d}{2}$ and $y=\frac{s-d}{2}$, therefore $xy=\frac{s+d}{2}\times\frac{s-d}{2}=\frac{ss-dd}{4}$.

239. PROBLEM 38. Given s and b, quære x and y.

$\left.\begin{array}{l} x+y=s \\ xx-yy=b \end{array}\right\}$ by the queſtion. From the firſt ſtep we have $x=s-y$, and $xx=ss-2sy+yy$, whence by the ſecond ſtep $ss-2sy=b$ and $y=\frac{ss-b}{2s}$, whence $x=s-\frac{ss-b}{2s}=\frac{2ss}{2s}-\frac{ss-b}{2s}=\frac{ss+b}{2s}$; that is, we have $x=\frac{ss+b}{2s}$, and $y=\frac{ss-b}{2s}$.

240. SYNTHESIS. $\frac{ss+b}{2s}+\frac{ss-b}{2s}=\frac{2ss}{2s}=s$

$\overline{\frac{ss+b}{2s}}\Big|^2=\frac{s^4+2ssb+bb}{4ss}$: again $\overline{\frac{ss-b}{2s}}\Big|^2=\frac{s^4-2ssb+bb}{4ss}$, whence their difference $\frac{+4ssb}{4ss}=b$.

241. When ss is leſs than b, then y is negative, and the firſt fundamental equation is changed into $x-y=s$ or more properly d; but the other fundamental equation remains the ſame; for the ſign of yy is not altered by changing the ſign of y. The problem is therefore changed into this; given d and b, quære x and y.

242.

242. *Example.* Let $s=10$ and $b=120$; then $ss=100$ and $\frac{ss+b}{2s}=\frac{100+120}{20}=\frac{220}{20}=11$; again $\frac{ss-b}{2s}=\frac{100-120}{20}=\frac{-20}{20}=-1$. So the two numbers are 11 and -1; that is, instead of the sum we are to take the difference of 11 and 1, which gives 10; but we are still to take the difference of their squares, which we shall find to be 120, as before prob. 10.

243. PROBLEM 39. Given s and q, quære x and y.

$\left.\begin{array}{l} x+y=s \\ \frac{x}{y}=q \end{array}\right\}$ by the question. From the second equation we have $x=qy$, therefore from the first equation $qy+y=s$, and $y=\frac{s}{q+1}$; now $x=qy=\frac{qs}{q+1}$, or $x=\frac{qs}{q+1}$ and $y=\frac{s}{q+1}$.

244. SYNTHESIS. $\frac{qs}{q+1}+\frac{s}{q+1}=\frac{qs+s}{q+1}=\frac{\overline{q+1}\times s}{q+1}=s$. Again $\frac{qs}{q+1}$ divided by $\frac{s}{q+1}=\frac{qs}{q+1}\times\frac{q+1}{s}=\frac{q+1}{q+1}\times\frac{s}{s}\times q=q$.

245. PROBLEM 40. Given p and q, quære x and y.

$\left.\begin{array}{l} xy=p \\ \frac{x}{y}=q \end{array}\right\}$ by the question. From the second equation $x=qy$, whence, and from the first equation, $qyy=p$, whence $yy=\frac{p}{q}$, and $y=\sqrt{\frac{p}{q}}$. Now because $x=qy$, therefore $xx=qqyy$. For yy, put its value before found, and $xx=qq\times\frac{p}{q}=pq$, therefore $x=\sqrt{pq}$.

246.

246. SYNTHESIS. $xx=pq$, and $yy=\frac{p}{q}$, therefore $xxyy=pq\times\frac{p}{q}=pp$, therefore $xy=p$. Again $\frac{xx}{yy}=pq$ divided by $\frac{p}{q}=\frac{pq}{1}\times\frac{p}{q}=qq$, whence $\frac{x}{y}=q$.

247. PROBLEM 41. Given s and p, quære x and y.

$\left.\begin{array}{r}x+y=s\\ xy=p\end{array}\right\}$ by the queſtion. From the firſt equation we have $y=s-x$; ſubſtitute this for y in the ſecond equation, and we have $x\times\overline{s-x}=p$, therefore $sx-xx=p$, which is an adfected quadratic equation; therefore by proceſs 2d. $xx-sx=-p$; by proceſs 3d. $xx-sx+\frac{ss}{4}=\frac{ss}{4}-p=\frac{RR}{4}$; by proceſs 4th. $x-\frac{s}{2}=\pm\frac{R}{2}$, and $x=\frac{s\pm R}{2}$, and $y=\frac{s\mp R}{2}$. See problem 27.

Now we had $\frac{ss}{4}-p=\frac{RR}{4}$. Multiply both ſides by 4, and $ss-4p=RR$; therefore make $ss-4pp$ equal to RR, and x and y are $\frac{s\pm R}{2}$.

248. It appears from this ſolution, that if $4p$ be greater than ss, the problem is impoſſible: on the contrary, in all poſſible caſes (that is, where x and y are any real numbers) ss is greater than $4p$, or $\frac{ss}{4}$ greater than p, or $\frac{s}{2}$ greater than $\sqrt{p}$, or $\frac{x+y}{2}$ greater than $\sqrt{xy}$. Or the arithmetical mean (as it is called) greater than the geometrical mean.

249. It will be convenient now to introduce a new character, to wit $>$, which ſtands for *greater than*, thus $\frac{s}{2}>\sqrt{p}$, means, half of s is *greater than*, the ſquare root of p.

250.

250. Because $ss-4p=RR$, therefore $ss>RR$, and $s>R$; consequently $\frac{s-R}{2}$ is affirmative. That is, both x and y are always affirmative.

251. SYNTHESIS. $\frac{s+R}{2}+\frac{s-R}{2}=\frac{2s}{2}=s$.

$\frac{s+R}{2}\times\frac{s-R}{2}=\frac{ss-RR}{4}=\frac{ss}{4}-\frac{RR}{4}=\frac{ss}{4}-\sqrt{\frac{ss}{4}-p}=p$.
Otherwise for $-RR$ substitute its value, which is $-ss+4p$, and $\frac{ss-RR}{4}=\frac{ss-ss+4p}{4}=\frac{4p}{4}=p$.

252. PROBLEM 42. Given d and p, quære x and y.
$\left.\begin{array}{r}x-y=d\\ xy=p\end{array}\right\}$ by the question. From the first equation we have $x=y+d$, and from the second $yy+dy=p$; therefore by process 3d. $yy+dy+\frac{dd}{4}=p+\frac{dd}{4}=\frac{RR}{4}$.
By process 4th. $y+\frac{d}{2}=\frac{\pm R}{2}$, $y=\frac{\pm R-d}{2}$.

Taking $y=\frac{+R-d}{2}$, we shall have $x=\frac{R+d}{2}$.

Taking $y=\frac{-R-d}{2}$, we shall have $x=\frac{d-R}{2}$.

253. In this latter case both x and y are negative; that y is so, needs no proof; that x is so will appear, because R is greater than d. For, because $p+\frac{dd}{4}=\frac{RR}{4}$, therefore $4p+dd=RR$: therefore $RR>dd$, and $R>d$. As x and y are always either both affirmative or both negative, $x-y$ will always represent their difference ($x-y$, when both are negative, changing to $-x+y$); and the solution of this problem cannot be made to include that of the former. Nor can the former problem be made to include this; for x and y there, are always affirmative.

254.

254. Hence this problem is always poſſible, whatever numbers be aſſumed for the difference and product of the unknown numbers, becauſe RR will always be affirmative, even though d be negative.

255. SYNTHESIS. $\frac{R+d}{2} - \frac{R-d}{2} = \frac{2d}{2} = d$.

Alſo $\frac{d-R}{2} - \frac{-d-R}{2} = \frac{2d}{2} = d$.

Again $\frac{R+d}{2} \times \frac{R-d}{2} = \frac{RR-dd}{4}$. For RR put its value, and we have $\frac{\overline{4p+dd}-dd}{4} = \frac{4p}{4} = p$.

Alſo $\frac{d-R}{2} \times \frac{-d-R}{2}$ gives $\frac{RR-dd}{4}$, as before.

256. PROBLEM 43. Given a and p, quære x and y.

$\left.\begin{array}{l} xy=p \\ xx+yy=a \end{array}\right\}$ by the queſtion. From the firſt equation $y=\frac{p}{x}$, and $yy=\frac{pp}{xx}$; therefore by the ſecond equation $xx+\frac{pp}{xx}=a$, and $x^4+pp=axx$, whence $x^4-axx = -pp$.

Now this is a biquadratic equation, but in the form of a quadratic equation, and may be ſolved in like manner. Thus we may conſider the unknown ſide of the equation as two terms out of three that make the ſquare of a binomial. Therefore compleating the ſquare, we have $x^4-axx+\frac{aa}{4}=\frac{aa}{4}-pp=\frac{RR}{4}$. By proceſs 4th. $xx-\frac{a}{2}=\pm\frac{R}{2}$. By proceſs 5th. $xx=\frac{a\pm R}{2}$, and $x=\pm\sqrt{\frac{a\pm R}{2}}$. If we ſuppoſe $x=\sqrt{\frac{a+R}{2}}$, then $y=\sqrt{\frac{a-R}{2}}$, becauſe the fundamental equations are the ſame, whether we ſuppoſe

x the

x the greater or the less of the two numbers (see par. 186.): because $\frac{aa}{4}-pp=\frac{RR}{4}$, therefore $aa-4pp=RR$.

257. SYNTHESIS. Because $xx=\frac{a+R}{2}$, and $yy=\frac{a-R}{2}$, therefore $xx+yy=\frac{a+R}{2}+\frac{a-R}{2}=\frac{2a}{2}=a$.

Again, $xx\times yy=\frac{a+R}{2}\times\frac{a-R}{2}=\frac{aa-RR}{4}=\frac{aa-\overline{aa-4pp}}{4}=\frac{4pp}{4}=pp$; but if $xxyy=pp$, extract the square root on both sides, and we have $xy=p$.

258. Hence the problem is impossible when $4pp$ is greater than aa, or $2p>a$, or $p>\frac{1}{2}a$. Moreover, as RR is always affirmative (in all possible cases) and equal to $aa-4pp$; therefore $aa>RR$, and $a>R$. Consequently $\frac{a-R}{2}$ is always affirmative, and therefore yy affirmative; that is, when x is possible, y is possible also: and both x and y must be affirmative.

259. PROBLEM 44. Given a and s, quære x and y.

$\left.\begin{array}{l}x+y=s\\xx+yy=a\end{array}\right\}$ by the question. From the first equation $y=s-x$, and $\overline{yy=ss-2sx+xx}$. From the second equation $xx+ss-2sx+xx=a$, that is, $2xx-2sx+ss=a$, and $2xx-2sx=a-ss$: whence by process 2d. $xx-sx=\frac{a-ss}{2}$. By process 2d. $xx-sx+\frac{ss}{4}=\frac{a-ss}{2}+\frac{ss}{4}=\frac{RR}{4}$. By process 4th. $x-\frac{s}{2}=\frac{\pm R}{2}$, and $x=\frac{s\pm R}{2}$.

260.

260. Here if $x = \frac{s+R}{2}$, y will $= \frac{s-R}{2}$; and contrariwiſe, as has been before obſerved.

Becauſe $\frac{a-ss}{2} + \frac{ss}{4} = \frac{RR}{4}$, therefore $\frac{2a-2ss}{4} + \frac{ss}{4} = \frac{RR}{4}$, or $\frac{2a-ss}{4} = \frac{RR}{4}$, or $2a - ss = RR$. Therefore the problem becomes impoſſible, when ss is greater than $2a$. In all poſſible caſes therefore $ss < 2a$. Let us now enquire, whether y may not be negative, or $s > R$. Now if $s > R$, then $ss > RR$, or $ss > 2a - ss$ or $2ss > 2a$, or $ss > a$. We ſee then in the firſt place, that ss muſt be leſs than $2a$, to make the problem poſſible; but, ſecondly, ss may alſo be leſs than a, and then y or $\frac{s-R}{2}$ will be negative, while x or $\frac{s+R}{2}$ continues affirmative: that is, the firſt fundamental equation becomes $x - y = s$, or rather $= d$. But the ſecond fundamental equation, or $xx + yy = a$, continues as before (for the ſquare of $-y$ is $+yy$). The problem then is now changed into this: Given the difference of two numbers and the ſum of their ſquares, to find the number. Or rather the problem ſhould be propoſed more generally ſo as to include both, thus: What two numbers are thoſe, which *incorporated* together make s, and the ſum of whoſe ſquares is a? See par. 203,

261. *Example.* Let the two numbers when incorporated make 20, and let the ſum of their ſquares be 250. Here then $s = 20$, $ss = 400$; alſo $a = 250$ and $2a = 500$. Here ss is leſs than $2a$ (for $400 < 500$, therefore the problem is poſſible. Again $ss > a$ (for $400 > 250$), therefore 20 will be the *arithmetical* ſum of the two numbers, not their difference. Now $2a - ss = 500 - 400 = 100 = RR$, and $R = 10$, and $\frac{s+R}{2} = \frac{20+10}{2} = \frac{30}{2} = 15$, and $\frac{s-R}{2} = \frac{20-10}{2} = \frac{10}{2} = 5$.

262. *Again*; Let the two numbers when incorporated make 10, and the ſum of their ſquares, as before, be 250. Then $s=10$, $ss=100$, $a=250$, $2a=500$. Here again, becauſe $ss<2a$ (for $100<500$), the problem is poſſible; but becauſe ss is alſo leſs than a, or $100<250$, therefore y will be negative. For $2a-ss$, or $500-100=400=RR$, and $R=20$, and $\frac{s+R}{2}=\frac{10+20}{2}=\frac{30}{2}-15$, and $\frac{s-R}{2}=\frac{10-20}{2}=\frac{-10}{2}=-5$: therefore $x=15$, and y is -5. That is, 5 taken from (*not added to*) 15 makes 10, and the ſquare of 5 added to the ſquare of 15 make 250.

263. If $ss=2a$, then is $R=0$, and the two numbers x and y are each equal to $\frac{1}{2}s$. If $ss=a$, then is $2a-ss$, or $RR=2ss-ss=ss$, and $R=s$, and $\frac{s+R}{2}=\frac{s+s}{2}=\frac{2s}{2}=s$; that is, $x=s$, and of courſe $y=0$, and $xx+yy$ is $=ss+0$.

264. SYNTHESIS. $\frac{s\pm R}{2}+\frac{s\mp R}{2}=\frac{2s}{2}=s$.

$$\left.\begin{array}{l} \left(\frac{s\pm R}{2}\right)^2=\frac{ss\pm 2sR+RR}{4} \\ \left(\frac{s\mp k}{2}\right)^2=\frac{ss\mp 2sR+RR}{4} \end{array}\right\}\text{The ſum of theſe two is}$$

$\frac{2ss \;*\; +2RR}{4}=\frac{ss+RR}{2}$. For RR write its value; and this ſum is $\frac{ss+2a-ss}{2}=\frac{2a}{2}=a$.

265. The following problems ſuppoſe the reader acquainted with the firſt ſix books of the Elements of Euclid. They are intended to give the learner ſome idea how algebra may be applied to the ſolution of geometrical problems, and to ſhew the nature and uſe of geometrical conſtructions; without entering formally into ſo difficult and extenſive a ſubject, as what is uſually underſtood by the *application of algebra and geometry to each other*.

266. PRO-

266. PROBLEM 45. In the right angled triangle ABC (fig. 1.) given the perpendicular $AC=\sqrt{b}$, and the ſum of the hypothenuſe and baſe, or $CB+BA=s$, to find the triangle; that is, to find the hypothenuſe and baſe ſeparately.

Call the hypothenuſe x, and the baſe y, and from the terms of the queſtion $x+y=s$, and from the nature of the figure (47. I. El.) $xx-yy=b$. The two fundamental equations then are the ſame as in problem 38. Therefore by that problem $x=\frac{ss+b}{2s}$ and $y=\frac{ss-b}{2s}$.

267. GEOMETRICAL CONSTRUCTION. Let AC be the given perpendicular. Through the extremity A, draw AS at right angles to it (11. I. El.) and equal to s the propoſed ſum. Join CS, and biſect it in E (10. I. El.) through E draw EB perpendicular to CS (11. I. El.) cutting AS in B. Join CB, and ABC will be the triangle required.

Demonſtration For the right angled triangles CEB. SEB, having $CE=SE$, and BE common, are equal, (4. I. El.) and BC will be equal to BS. Therefore $AB+BC=AB+BS=AS=s$.

268. This conſtruction affords the very ſame calculation that the algebraic proceſs does. For the triangles ASC and ESB, having each a right angle, and the angle at S common, will alſo have the remaining angle equal, (Cor. to 32. I. El.); therefore being ſimilar we have (by 4. VI. El.).

$$SA : SC :: SE\ (\text{or } \tfrac{1}{2}SC) : SB=\frac{\overline{SC}|^2}{2SA}:$$

but $\overline{SC}|^2=\overline{SA}|^2+\overline{AC}|^2=ss+b$;

therefore SB or $CB=\frac{ss+b}{2s}$, as before.

One great uſe of geometrical conſtructions, is to draw out a rule for calculating the unknown quantities from the known ones; alſo to determine their

limits, &c. as we may ſee in this and the ſubſequent problems.

269. It is ſometimes of uſe to reduce the queſtion to a problem in trigonometry. The trigonometrical tables, ready calculated, are thus made ſubſervient to the finding the unknown quantity arithmetically. Great uſe is made of thoſe tables, and the tables of logarithms in more difficult problems.

270. TRIGONOMETRICAL CALCULATION. In the right angled triangle *CAS*, we have the ſides *AC* and *AS* given, whence by trigonometry we get the angle *ASC*; conſequently *ABC*, which is equal to twice *ASC* (5 and 32. I. El.), whence we have *ACB*.

Therefore in the right angled triangle, having one ſide and all the angles given, the other two ſides may be found by the trigonometrical tables.

271. We took notice in problem 38, that when *ss* is leſs than *b*, then *y* is negative. Correſponding to this in the geometrical conſtruction, when $AS = AC$ (fig. 2.) the point *B* falls upon *A*. For the triangle *CAS*, being in that caſe iſoſceles, the line *EB*, which biſects the baſe perpendicularly, paſſes through the vertex (12. I. El.). If *AS* be taken ſtill leſs, the point *B* will fall on the other ſide of *A*, that *S* does, and *AB* will be negative (fig. 3.). *AS*, which before (in fig. 1.) was the ſum of *BA* and *BS*, will now (in fig. 3.) be their difference; and we ſhall find a right angled triangle, having the ſame given perpendicular *AC* as before, but having the *difference* of the hypothenuſe and baſe, or $CB - CA$, equal to the given line *AS*, or *s*.

272. PROBLEM 46. To divide a given line *AB* (fig. 4.) into two parts *AE* and *EB*, ſo that the rectangle under the parts *AE* aud *EB* may be equal to a given ſquare *FGHI*.

Call the given line $AB = s$, the given ſquare *FGHI* $= p$, conſequently its ſide $= \sqrt{p}$. Call the parts *AE* and *EB*, *x* and *y*: then by the terms of the queſtion

$x+y=s$, alſo $xy=p$; whence the algebraic ſolution is the ſame as that of problem 41.

273. Geometrical Construction. On the given line AB, as a diameter, deſcribe a ſemicircle; on the ſame line produced take FG, equal to the ſide of the propoſed ſquare, and on it conſtruct a ſquare $FGHI$, equal to the propoſed ſquare (46. I. El.): produce the ſide HI (oppoſite to FG) till it meets the ſemicircle in D and d; then if perpendiculars DE and de be let fall from the points D and d (12. I. El.) either of them will divide the line AB in the manner required. For $AE\times EB=\overline{DE}|^2$ or $Ae\times eB=\overline{ae}|^2$ (35. III. El.).

Let C be the center of the ſemicircle, and ſince $DE=de$ by conſtruction, we have $CE=Ce$, whence $AE=eB$ and $EB=Ae$; and the two anſwers are alike, as was obſerved, problem 27. par. 186.

274. From this conſtruction, we get the following method of calculation. Draw the radius $CD=\frac{s}{2}$, and (by 47. I. El.) $\overline{CD}^2-\overline{DE}^2=\overline{CE}^2$; that is, $\frac{ss}{4}-p$, or $\frac{ss-4p}{4}$, or (by ſubſtitution) $\frac{RR}{4}=\overline{CE}^2$, and $\frac{R}{2}=CE$, whence $AC+CE$ or $AE=\frac{s}{2}+\frac{R}{2}=\frac{s+R}{2}$: alſo $CB-CE$ or $EB=\frac{s}{2}-\frac{R}{2}=\frac{s-R}{2}$; that is, make $ss-4p=RP$, and the two parts AE and $EB=\frac{s\pm R}{2}$, as in problem 41.

275. If the ſide of the ſquare $FG=FH$ is equal to the radius of the ſemicircle, that is, if $\sqrt{p}=\frac{s}{2}$, the line IHD will touch the circle only; the two points D and d will coaleſce into one; the line CE vaniſh, and the two parts AE and EB become equal to each other, and to half the whole line AB. If the ſide of the

 ſquare

ſquare be taken ſtill greater, the line *IH* will fall wholly without the circle, and the problem become impoſſible.

276. The points *AEB* never change their order; therefore the algebraic ſigns of the two parts *AE* and *EB* will never be changed, from whatever point we date the origin of theſe two lines *AE* and *EB*; whether from *A*, one extremity of the given line *AB*, or from the point *E*, which ſeparates the two parts *AE* and *EB*.

277. TRIGONOMETRICAL CALCULATION. In the right angled triangle *CED* we have the hypothenuſe $CD = \frac{1}{2}s$, and the perpendicular $DE = \sqrt{p}$, whence the angle *DCE*, and its coſine *CE* to radius *CD*, and of conſequence the parts *AE* and *EB*.

278. *Scholium.* This problem is the ſame as problem 41. viz. Given the rectangle under two quantities; for inſtance, two lines and their ſum, to find thoſe lines. It might be expected that this problem, like the laſt, ſhould include alſo another problem, viz. "Given the rectangle under two lines and their *difference*, to find thoſe lines," but it does not; for the algebraic ſigns of the lines *AE* and *EB* are always the ſame; therefore the other problem muſt have a ſeparate ſolution as follows.

279. PROBLEM 47. Given the difference between the (adjacent) ſides of a right angled parallelogram $= d$, and its area equal to that of a ſquare, whoſe ſide is $\sqrt{p}$; to find the ſides ſeparately.

Call the two ſides x and y, and by the terms of the queſtion $x - y = d$, and $xy = p$, whence the algebraic ſolution is the ſame as that of problem 42.

280. To CONSTRUCT this problem geometrically, let *AB* (fig. 5.) be the propoſed difference. Erect *BS* perpendicular to it (11. I. El.) and equal to $\sqrt{p}$, the ſide of the propoſed ſquare. On *AB*, as a diameter, deſcribe a circle, and through its center *C* draw

draw SEF, cutting the circle in E and F, and SE and SF will be the two ſides of the parallelogram.

For $SF - SE = FE = AB =$ the propoſed difference; and becauſe SB is a tangent to the circle in B (16. III. El.) therefore $SE \times SF = \overline{SB}^2$ (36. III. El.).

281. Algebraic Calculation. Becauſe CBS is a right angled triangle, therefore $\overline{BS}^2 + \overline{BC}^2 = \overline{CS}^2$, that is, $\frac{dd}{4} + p$, or $\frac{dd + 4p}{4}$, or $\frac{RR}{4} = \overline{CS}^2$, therefore $\frac{R}{2} = CS$, and $CS + CF$, or $SF = \frac{R}{2} + \frac{d}{2} = \frac{R+d}{2}$, and $CS - CE$, or $SE = \frac{R}{2} - \frac{d}{2}$ or $\frac{R-d}{2}$, as in problem 42.

The line SF being drawn through the center of the circle, will always cut it in two points; therefore the problem will be always poſſible, and the points $SECF$ always lie in the ſame order; therefore the lines SF and SE have the ſame algebraic ſign, their common origin being at S.

282. Problem 48. In a right angled triangle, given the hypothenuſe AB (fig. 6.) $= \sqrt{a}$, and the area, equal to half the ſquare, whoſe ſide is $\sqrt{p}$: to find the triangle.

Call the perpendicular and baſe x and y, and (by 47. I. El.) $xx + yy = a$: moreover xy is double the area of the triangle (41. I. El.) therefore $xy = p$; whence the algebraic ſolution will be the ſame as that of problem 43.

283. Sometimes the conditions of a problem are ſo independent one of another, that each may be conſtructed ſeparately. One condition (as we have ſaid) does not confine the problem to a ſingle anſwer when there are two unknown quantities. In conſtructing one condition then, we are not to find one particular line of a determinate length; but all thoſe lines which are of ſuch a length as will anſwer that one condition. If the extremities of all theſe lines are joined; they

will form either a curve or a right line, which is called the *locus* of that condition. If in like manner the *locus* of the other condition be drawn, it is manifest, the intersection of these two *loci*, will determine the problem. For the point of intersection being common to both *loci* will satisfy both conditions. All this will be understood by the following construction of the problem before us.

284. The first condition is, that the triangle is right angled, having AB for its hypothenuse; therefore upon AB, as a diameter (fig. 6.), describe a semicircle, $AdDB$: now, as all angles in a semicircle are right angles (31. III. El.), this semicircle will be the locus of the vertexes of all the right angled triangles that can be described on the given hypothenuse AB. In other words, the vertex of every right angled triangle (on that side of the diameter AB) having AB for its hypothenuse, will be in the arch $AdDB$. The other condition is, that the area of this triangle is equal to half the square whose side is $\sqrt{p}$. To construct this condition, on AB, as a base, describe a right angled parallelogram $ABHI$ equal to the proposed square (45. I. El.), or double the area of the triangle; produce the side HI, and it will be the locus of the vertexes of all those triangles, whose base is AB, and which have the proposed area (37. I. El.); therefore the two intersections of the right line HI and the semicircle $AdDB$, will either of them determine the vertex of the triangle sought.

285. The problem becomes impossible when AH is greater than the radius of the semicircle, or AH greater than $\frac{1}{2}AB$, or when $AH \times AB > \frac{1}{2}AB \times AB$, or when $p > \frac{1}{2}a$.

286. TRIGONOMETRICAL CALCULATION. Draw the radius CD, and let fall the perpendicular $DE = BI$, and $AC \times CB : AC \times DE :: CD : DE$; but $AC \times CB = \overline{AC}$

$\overline{AC}^2$, and $AC \times DE = \frac{1}{2} AB \times DE =$ the area of the triangle; therefore we have this analogy. As the ſquare of half the hypothenuſe is to the area of the triangle (or as the ſquare of the whole hypothenuſe, is to four times the area of the triangle), ſo is CD, the radius, to DE, the ſine of the angle DCE, which is double the angle DAC (20. III. El.). The hypothenuſe AB of the right angled triangle ADB being given, and one of the acute angles DAB being found as before, the legs AD and DB will be found by the common rules of Trigonometry.

287. PROBLEM 49. In a right angled triangle (fig. 7.) given the hypothenuſe $AB = \sqrt{a}$, and the ſum of the other two ſides $= s$; to find the triangle.

Call the other two ſides x and y, and by the condition of the problem $x + y = s$, and (by 47. I. El.) $xx + yy = a$, whence the algebraic ſolution will be the ſame as that of problem 44.

288. This problem may alſo be conſtructed by means of loci, thus: On the hypothenuſe AB, as a diameter, deſcribe a ſemicircle $AdDB$, and it will be the locus of the vertexes of all the right angled triangles that can be deſcribed on the given hypothenuſe, as before. Again; From the center C, on AB produced, ſet off, CH and CI both ways each equal to half s; then on the axis HI, and with the foci A and B, deſcribe a ſemi-ellipſe, and it will be the locus of the vertexes of all thoſe triangles, that have the ſum of their ſides AD and $DB = HI$ or s. For from the nature of the ellipſe, if two lines AD and BD, be drawn from the two foci A and B to any point D in the ellipſe, their ſum will be equal to the axis HI.

289. But it is a rule, that no problem ſhould be conſtructed by means of the conic ſections, that can be conſtructed by means of the circle and right lines only: becauſe the deſcription of the circle is ſimpler than that of the conic ſections; the properties of the circle and right lined figures are better known, and their

their relations easier discerned than those of the conic sections. Hence constructions, in which the circle and right lines only are concerned, are best fitted to determine the limits of a problem, furnish a method of calculation, and other purposes of a construction. In the present case, it is easy to see the problem becomes impossible when the ellipse passes wholly over the circle. But it is not easy to see what will be the proportion of *HI*, the axis of the ellipse, to *AB*, the diameter of the circle when that happens. Nor does there appear, from this construction, any method of calculating the value of *AD* and *BD*, the sides sought, without further principles than what are contained in Euclid's Elements.

We may construct the problem, by a circle, and right line, in manner following.

290. In the right angle *IAK* (fig. 8.) with the center *A* and radius *AB*, equal to the given hypothenuse, describe the arch of a circle *BDdF*. On *AB* and *AF*, take *AI* and *AK* equal to *s*, the sum of the sides given. Join *KI* by a line intersecting the arch of the circle in *D* and *d*; draw the radii *AD* and *Ad*, let fall the perpendiculars *DE* and *De*, from the points *D* and *d* on the base *AI* (12. I. El.), and either of the triangles *ADE* or *Ade* will answer the conditions required. For (by 29. I. El.) the angle $EDI = AKI = EID$ (by 5. I. El.); therefore (by 6. I. El.) *DE* is always equal to *EI*, therefore $AE + ED$, or the sum of the sides equal $AE + EI = AI = s$. Moreover $AD = AB =$ the proposed hypothenuse (Definition 15. I. El.) therefore the right angled triangle *AED* will have its hypothenuse $AD = \sqrt{a}$, and the sum of its sides $AE + ED = s$.

291. The line *KI* may be considered as the locus of the vertexes of all those right angled triangles, the sum of whose legs is *AI*, and whose base is in the line *AI*; the circle *BDdF* is manifestly the locus of the vertexes of all those right angled triangles, whose hypothenuse is *AD*, the radius of the circle; therefore the

the interſection of the right line and circle will determine the triangle ſought.

292. Hence this TRIGONOMETRICAL method of CALCULATION. In the triangle *ADI* there is given the ſide $AI = s$, the ſide $AD = \sqrt{a}$, and the angle $AID = 45$ degrees, whence the angle *DAI* and the ſide *DI* may be found, and thence *DE* and *AE*.

293. Let *IK* (fig. 9.) touch the circle in *T*, then *ATI* is a right angle (18 III. El.), therefore $TIA = TAI$ (32. I. El.) and $AT = TI$ (6. I. El.), therefore $\overline{AI}^2 = AT^2 + TI^2$ (47. I. El.) $= 2\overline{AT}^2$, that is, $ss = 2a$; therefore if *s* be ſtill greater, that is, $ss > 2a$, the line *IK* will fall wholly without the circle, and the problem becomes impoſſible

294. In the two foregoing problems, 47 and 48, it appeared, that *x* being affirmative, *y* could never become negative (ſee alſo prob. 42 and 43), but the contrary may be the caſe in this; becanſe *s* may be leſs than *R*; ſee problem 44. And the geometrical conſtruction anſwers to all that was obſerved of the algebraic proceſs in problem 44; as ſhall now be ſhown.

295. In the geometrical conſtruction *AI* was taken equal to the ſum of the legs of the right angled triangle; let now *AI* diminiſh ſo as to be equal to $AB = AD =$ the hypothenuſe (fig. 8.), that is, let $s = \sqrt{a}$, and the points *E* and *I*, coinciding in *B*, alſo the points *Kd*, coinciding in *F* (fig. 10.) the triangles *ADE* and *Ade* both vaniſh; the problem in that parcicular caſe becoming impoſſible (20. I. El.). If the line *AI* ſtill diminiſhes, ſo that *AI* becomes leſs than *AB* or $s < \sqrt{a}$, then the point *I* will fall on the contrary ſide of the point *E*, and the line *EI* becomes negative (fig. 11.) and *AI* inſtead of the ſum, will now become the difference of *AE* and *EI*; becauſe $DE = EI$, and $de = eI$ (as before) either of the right angled triangles *ADE* or *Ade*, will have the difference of their legs equal

to

to the given line AI, and $AD = AB$ the propoſed hypothenuſe.

296. We may here remark, that in all theſe changes, the line KI ever moves parallel to itſelf, or rather to the line BF.

297. If we purſue theſe changes further, and ſuppoſe the difference of the legs ſtill to decreaſe, and at laſt to vaniſh, we ſhall have in the algebraic part d, (which was s) now $=0$. $RR = 2a$, and x and y both equal to $\frac{R}{2} = \sqrt{\frac{RR}{4}} = \sqrt{\frac{1}{2}a}$. In the geometrical conſtruction (fig. 12.) the lines AI and AK vaniſhing, the points K and I coincide in A. Nevertheleſs, if we continue the parallel motion of the line KI till it paſſes through A, it determines the triangles AED and Ade, and we ſhall have the legs AE and ED equal to each other and to the ſquare root of $\frac{1}{2}AD$, or $\sqrt{\frac{1}{2}a}$.

298. If we ſtill continue this parallel motion of the line KI, we ſhall again have (fig. 13.) AI equal the difference of AE and EI: but now lying the contrary way, it ſhows that their difference (or the quantity $AE - EI$) is now negative. AE, which before was the greater leg of the triangle, being now become the leſs. In the algebraic proceſs, the equation $x - y = d$, now becomes $x - y = -d$, $RR = 2a - dd$, as before. $x = \frac{R-d}{2}$, and $y = \frac{R+d}{2}$.

299. This will be the caſe till the points K and D coinciding, the triangles ADE and Ade, do again vaniſh. Laſtly, the parallel motion of the line KI being yet further continued, the points A, E, I, will will lie in the ſame order as at firſt, but on contrary ſides of the center (fig. 14.) and AI will again be the ſum of AE and EI as at firſt; the lines AE and EI being

being negative, the algebraic equation will now be $-x-y=-s$, and $xx+yy=a$. Here s is negative, but the quantity a is never negative; and correſponding to this, the length of the radius AD, being always computed from A the center, never becomes negative.

300. We have thought it uſeful to purſue theſe changes through every poſſible caſe; to ſhow how ſtrict the analogy is between algebraic equations and geometrical conſtructions.

AN INTRODUCTION AND REMARKS ON SEVERAL PROPOSITIONS IN THE FIRST SIX BOOKS OF EUCLID, AS THEY STAND IN DR. SIMSON'S TRANSLATION.

AN

INTRODUCTION

TO

EUCLID's ELEMENTS.

1. THE businefs of fcience is from a few general principles to draw a great number of particular conclufions; therefore the propofitions in which thofe principles are delivered, muft contain very general ideas. As general ideas are not the objects of fenfe, but are formed by that operation of the mind we call *abftraction* (Locke, B. I. C. XI. S. 9.) no wonder the firft principles of all fcience fhould with difficulty be underftood by beginners.

2. There are thofe who advife beginners in Geometry, not to be over folicitous about underftanding the definitions perfectly, alledging that the meaning of the terms will be learned by the ufe of them in the propofitions. And it is undoubtedly true, that we really get moft of our abftract ideas, not by formal definitions, but by obferving the ufe and force of words in common language; by obferving to what a number of objects, each differing in fome particulars (though always agreeing in fome others) the fame word is applied.

3. It may be ufeful here, to fhow more particularly what thefe abftract ideas are, and how we come by them, as laid down by Mr. Locke, and apply his doctrine to the definitions of Euclid.

K 4. Mr.

4. Mr. Locke divides the ideas in our minds into Simple and Complex. Simple ideas being uncompounded, are originally received either by the ſenſes, or from reflecting on what paſſes in our own minds; and in acquiring theſe the mind is wholly paſſive: it cannot create one new ſimple idea. There is no idea, the origin of which cannot be traced up to one of thoſe ſources; no idea but what is got originally either from ſenſation or reflection. But the mind once ſtored with a number of ſimple ideas, can exert ſeveral wonderful acts of its own upon them; ſo that out of theſe, as the materials, many other ideas are formed. The mind can *combine* ſeveral ſimple ideas together, into one compound; forming what is called a *complex* idea. When a *name* is given to an idea thus compounded of ſuch and ſuch ſpecific parts, it does as it were tie thoſe ſeveral parts together; ſo that when we hear that name pronounced, the intire compound (conſiſting of thoſe ſpecific parts and no other) does at once preſent itſelf to the mind. The mind alſo has a power of *diſcerning* or diſcriminating ideas, whether they be ſimple or complex ideas: it can diſcern in what particulars two ſuch ideas agree, and in what they diſagree. The mind alſo can *compare* two ideas with reſpect to extent, degree, time, place, and a number of other circumſtances. From this compariſon of ideas with one another, ariſes what is called, *Relation*. Among other relations, that which in mathematics is called *Proportion*, and which ariſes from comparing quantities together in reſpect of their *Magnitude* only.

5. But the moſt important power of the mind over its compound ideas, is that artificial ſeparation of them called ABSTRACTION. When we become acquainted with a variety of objects really exiſting, the ideas of which are all complex, we begin to remark, that the component parts of theſe complex ideas are often, in a great meaſure, the ſame, though in part different. We alſo find that we have occaſion to conſider theſe ideas and to ſpeak of them, only ſo far forth

forth as they agree; neglecting that part in which they differ. We therefore give a name to the assemblage of such of those ideas that are common to all these complex ideas, which name, as it were, ties those particular ideas together; and such an assemblage is called an *abstract*, or *general* idea. Now here it is evident, that such an abstract idea, is a mere creature of the mind, formed by its own power over its own ideas, and that an abstract idea, has no architype or real existence without, that corresponds to it; for out of a variety of complex ideas, each having a real architype without, we select parts, and therefore leave out parts; which parts so left out, are necessary to make a complex idea, corresponding to the architype or real existence without.

6. Thus the idea of every triangle really existing, and whose form we see delineated on paper, is different. One triangle has all its angles acute. Another is what is called a right angled triangle: another is an obtuse angled triangle. All acute angled triangles are not alike; but differ both with respect to their angles, and to the length of their sides. Every individual triangle actually existing has its own particular circumstances, both respecting its angles and its sides; and the complex idea of that particular triangle, is made up of the assemblage of all the particular ideas, excited by every one of these particular circumstances. All these circumstances occur particularly and specially in every triangle we see; and though we may give a name to an assemblage of two or three only of these particular ideas; yet such a partial assemblage has no architype without to correspond to it. Thus, in contemplating a variety of triangles drawn on paper, we may observe, that though their sides differ in length, and though their angles differ in acuteness, yet the three angles of every one of these triangles are each of them less than a right angle, or each of the three angles is acute. Observing that all these figures do thus agree in this one circumstance, we abstract from

thoſe circumſtances in which they differ (and the ideas they excite) and retain only the idea excited by this circumſtance in which they agree, and thus form in our minds the abſtract idea of an *acute angled* triangle.

7. In thus contemplating a variety of triangles of different forms, if we go further, and abſtract from all conſideration both of the magnitude of their ſides and of their angles; if we abſtract from the ideas which theſe particular circumſtances excite, and attend only to this ſingle circumſtance in every one of theſe figures (and the idea it excites), namely, that every one of theſe figures has three angles (neither more nor leſs in number) we then get the abſtract or general idea ſignified by the word *triangle*.

8. Yet this abſtract idea has no architype really exiſting. No one can draw the figure of a *triangle general*, a triangle which is neither acute-angled, right-angled, nor obtuſe-angled, and whoſe ſides are of no particular length. The general idea is a creature of the mind, and exiſts no where elſe.

9. In theſe general ideas, the circumſtances retained (or rather the ideas they raiſe) are to be found in every individual. Whatever therefore is proved in conſequence of theſe general ideas, muſt be true of every individual. Thus, if we form the general idea of a right lined figure, having three angles, and ſhow (in conſequence of it thus having three angles in number) that the ſum of thoſe three angles is exactly equal to two right angles; then this conſequence (proved of a triangle in general) is applicable to every particular triangle, whether acute, right-angled, or obtuſe, or whatever be the dimenſions of its ſides. It would be endleſs to make a proof of this, for every triangle of every ſort and ſize that exiſts, or can exiſt; but by thus forming general ideas, and reaſoning upon them, knowledge is extended to a vaſt number of particular ſubjects; and this makes what is called SCIENCE.

10. When we would convey to another an abſtract idea with which he is unacquainted; the beſt way (if

we can do it) is to trace out the progreſs of our own mind in forming it. We muſt begin with thoſe particular complex ideas, which have real architypes, point out the component parts of theſe complex ideas; ſhow which of them the mind rejects, and which it retains in forming the abſtract idea. Thus to explain the very abſtract idea, annexed to the word *point*, we muſt begin with the notion of a ſolid body; which idea our ſenſes will ſuggeſt in great variety of ways. We ſoon perceive that every ſolid body has length, breadth, and thickneſs. Now, if we exclude the idea of thickneſs, and retain thoſe of length and breadth, we ſhall get the abſtract idea of a ſurface; that is, the abſtract or mathematical idea; for a mathematical ſurface exiſts in idea only. All paper how thin ſoever, has yet ſome thickneſs, or it would be nothing. If from the original idea of a ſolid figure, conſiſting of length, breadth, and thickneſs, we exclude both breadth and thickneſs, and retain the idea of length, we get the idea of a mathematical line.

11. By further abſtraction, leaving out length, we get the very abſtract idea of a point; though I confeſs the operation of the mind, in this caſe, is ſo very ſubtile, that it can hardly be diſtinctly and clearly traced out. All abſtract ideas are mere creatures of the mind; they have no architypes without that ſtrike the ſenſes; they are objects of the underſtanding, not of the ſenſes; therefore they are difficult to be conceived; much more ſo very abſtracted an idea as that of a point. Well may learners, unaccuſtomed to mathematical and abſtract ideas, wonder what THAT is which has *no parts* and *no magnitude*, and which indeed has no real exiſtence any where but in the imagination only.

12. As abſtract ideas have no real exiſtences that anſwer to them, ſo Euclid's propoſitions are ideal only; they only ſhow the agreement or diſagreement of abſtract ideas. Whether there exiſts in nature ſuch a thing as a right line, may be doubted. The fineſt line

line drawn on paper, if viewed through a microſcope, would appear uneven, and of conſiderable breadth. But this is nothing to the purpoſe; for the agreement contended for, is between abſtract ideas, not real exiſtences. As far as an actually exiſtent right line, drawn on paper, agrees with the mathematical definition, ſo far Euclid's demonſtrations reſpect real exiſtences, and no further.

13. In tracing out the agreement of ideas, ſome are of ſuch a ſort, that their agreement is immediately perceived by *juxta-poſition* only; their agreement is perceived intuitively, and is confeſſed by all. Affirming ſuch an agreement is laying down an Axiom. The agreement of remote ideas is ſhown by the agreement of a train of intermediate ideas, each agreeing with that next to it: ſuch a train of ideas are called middle terms. When the agreement of the extreme ideas is thus inferred from the continual agreement of the intermediate ideas, this is called a Demonstration.

14. The agreement of ſome ideas muſt be original and intuitive. Demonſtration muſt be founded on axioms: nevertheleſs, the agreement of remote ideas once demonſtrated, is ever after to be taken for granted, and uſed in future demonſtrations as well as axioms.

15. The fewer the firſt principles are, from which the whole ſyſtem of geometry is derived, the more ſcientific is that ſyſtem: for this reaſon Euclid proves ſome truths (from his own firſt principles) which are undoubtedly of themſelves ſelf-evident. Nay, their own evidence carries a more forcible conviction of their truth than the demonſtration itſelf. Thus the fourth propoſition of the firſt book of the Elements, is thought by many to be more evident without the demonſtration than with it. So alſo the thirtieth of the firſt book, ſeems not to want a demonſtration; yet a very clear demonſtration is given by Euclid.

ON

ON THE DEFINITIONS.

16. In a definition, all those ideas are enumerated which constitute the abstract idea of the figure. Thus a triangle, is a right-lined figure having three angles (using the word *figure* in Euclid's sense, definition 14). Now this circumstance does necessarily imply many others: for instance, that the sum of the three angles, makes two right angles: this property is included in, and is virtually affirmed in the definition, and thus all the propositions in Euclid, are virtually included and contained in the definitions and axioms.

17. We said, that a definition was an enumeration of all the particular ideas, that constitute the whole complex idea. It follows from hence, that a simple idea cannot be defined. Another word may be used; he who does not know what the word *red* means, may understand the word *rouge*, but this is not laying down a definition. So the attempt to define a *straight line* (see Euclid, Def. 4.) is absurd. It is full as difficult to understand what is meant by a "*line's lying evenly between its extreme points*," as by its *being straight*. The same may be said of Euclid's definition of a plane *.

18. Dr. Simson has substituted another definition of a plane more intelligible, but not natural; it being (as himself says) a property of a plane superficies; not the idea of a plane superficies, nor what helps us to it. It must be observed, that in laying down this property, when mention is made of two points (see Simson's note on Def. 7.) it must be two points *wheresoever* taken. Thus in a cylindric superficies; two points may be taken in many parts of that curved superficies, and yet the line that joins them both be straight, and also wholly in that superficies; but two points may also be so taken, that this property will not hold true.

* A right line is only another name (derived from the Latin) for a straight line.

19. The idea ſignified by the word *angle* is alſo a ſimple idea. The *inclination* of two lines to each other (or the *opening* of two lines, as ſome call it) is only a periphraſis for an angle. A rectilineal angle may be diſtinguiſhed from a curvilineal angle, but the idea of the acuteneſs or obtuſeneſs of the angle itſelf, is a ſimple idea. The idea of a *term* (Def. 13.) is alſo a ſimple idea. Definition 19, occurs in definition 6, book the third, and is not wanted here.

20. We may therefore leave out definitions 4, 7, 8, 9, 13, 19. We may indeed retain definition 9, not as a definition, but as a periphraſis for the word *angle*. The technical word *angle* may be new to thoſe who have not read geometry, and the idea muſt be conveyed by a circumlocution of words in common uſe; but this is no more a definition, than the explaining the word *rouge* by *red*, or by *roſe-coloured*, is a definition of *rouge*.

21. We may remark on definition 10th, that it is the ideal equality of the two adjacent angles, which conſtitutes the ideal right angle, or, as practical mechanicians call it, a ſquare angle: not the look of it in the printed figure; nor the meaſuring it by the application of a joiner's ſquare. This ideal equality of the adjacent angles is ſometimes ſuppoſed, and is ſometimes to be demonſtrated.

22. We may remark on definition 31ſt, that ſuch a figure is alſo called a RECTANGLE. See Prop. 1. El. 2.

23. The definition of parallel lines (Def. 35.) is not very eaſy to be underſtood. Nor is the 12th axiom ſelf-evident: this is a ſubject of ſome difficulty, and has much exerciſed all the commentators. Dr. Simſon's judicious tranſlation of the Greek (avoiding the hackneyed and much abuſed term, *infinitely produced*) frees the reader from the difficulty and abſurdity of ſuppoſing, that a right line can be actually produced *in infinitum*, as it is called.

24. Mr. T. Simpſon, in the firſt edition of his uſeful book of Geometry, lays down this definition of

parallel

parallel lines: "Two right lines are ſaid to be parallel "or equidiſtant, when perpendiculars to one of them, "any where taken, and terminated by the other, are "equal, the one to the other." Yet in his ſecond edition he returns to Euclid's definition.

ON THE POSTULATES AND AXIOMS.

25. The Poſtulates are not to be ſo underſtood, as if Euclid required a practical dexterity in managing a ruler and pencil. They are here ſet down, that his readers may admit the *poſſibility* of what he may hereafter require to be done. To ſhow that what he thus requires contains no abſurdity, no repugnant ideas, Euclid in the courſe of a demonſtration requires you to produce a terminated ſtraight line. Was this as impoſſible in idea, as it is to take a greater number out of a leſs, the whole demonſtration muſt fail: for the ſteps of a demonſtration, like the links of a chain, hang by one another. Euclid therefore in this place enumerates all the operations required in his future demonſtrations and problems; requiring their poſſibility to be here acknowledged, and thus precludes all future objections on this head.

26. What an axiom and what a demonſtration is, we have before explained, par. 13. An axiom is a ſelf-evident truth; it requires no demonſtration: it may require to be explained, to be made intelligible; but as ſoon as underſtood, it ſhould be what all immediately aſſent to. The 12th axiom, as we have ſaid, is not properly an axiom (ſee Dr. Simſon's notes on it); but inſtead of it, you may ſubſtitute the following ſimple propoſition.

27. Axiom. Two ſtraight lines meeting in a point, are not *both* parallel to a third line.

Thus in fig. 24, the lines *AG* and *IG* meeting in *G*, are not both parallel to a third line *CD*.

In

In par. 42, this axiom is applied to the demonſtration of the 29th Prop. of I. El.*

28. In a geometrical propoſition, either ſomething is propoſed to be *done*, or ſome truth is to be *demonſtrated*. In the firſt caſe, the propoſition is called a PROBLEM; it the latter, it is called a THEOREM. What was ſaid of the poſtulates is applicable to the problems. They are to ſhow the poſſibility of doing ſomething, required to be done in the courſe of a demonſtration; to ſhew that what is required, is not abſurd and naturally impoſſible. In the poſtulates, the poſſibility of doing the thing, ſhould be ſelf-evident; in the problems, the poſſibility of doing the thing is proved, by laying down every ſtep that is neceſſary for the actual doing of it; each of which muſt be poſſible, and the whole ſuch as will effect the thing to be done.

29. The word *Propoſition*, in its general acceptation, is reſtrained to what Euclid calls a *Theorem*, in which the agreement between certain abſtract ideas is affirmed. In every ſuch propoſition, there is what the logicians call the *Subject* and the *Predicate*. So Prop. 37, viz. *Triangles on the ſame baſe and between the ſame parallels*, is the ſubject of this propoſition; and it is predicated of ſuch triangles, *that they are equal*. A *converſe propoſition*, is when the ſubject of the former propoſition becomes the predicate of the latter; and the predicate the ſubject. So in prop. 39, which is the converſe of prop. 37, "*Equal triangles on the ſame baſe* (Euclid adds, *on the ſame ſide*)" is the ſubject, and it is predicated of ſuch, *that they are between the ſame* pa-

* Straight lines that meet in a point are ſaid to have an inclination to each other. Such lines as being produced ever ſo far do not meet, may be ſaid to have no inclination to each other. See Def. 9 and 25. From this conſideration, perhaps, the axiom may be evident. It muſt be owned that Euclid's 12th axiom is better connected with his definition of parallel lines than this is. But then the demonſtration of Euclid's axiom, both in Clavius and in Simpſon, is very tedious.

parallels. So prop. 27, "*Two ſtraight lines having another falling on them, and making the alternate angles equal,*" is the ſubject, and it is predicated of thoſe two lines, "*that they are parallel.*" In prop. 29, "*Two parallel lines*" is the ſubject, and it is predicated of them, "*that if another falls upon them, the alternate angles will be equal.*" Therefore theſe are converſe propoſitions.

30. *Converſe* and *contrary* propoſitions are by no means to be confounded, as is commonly done; the contrary propoſition, is when what is affirmed in the former propoſition, is denied in the latter; but the ſubject and predicate in each is the ſame. Thus the contrary of prop. 37, is this: triangles on the ſame baſe and between the ſame parallels, are *not* equal to one another. The ſubject is ſtill, "Triangles on the ſame baſe and between the ſame parallels;" but now it is predicated of ſuch, that they are not equal, or that they are unequal.

31. Converſe propoſitions are not neceſſarily true, but require a demonſtration; and Euclid always demonſtrates ſuch as he has occaſion for. An inſtance or two will ſhow this. If two right lined figures are ſo exactly of a ſize and form (both reſpecting their ſides and angles) that being laid one on the other, their boundary lines do exactly coincide and agree; then no one doubts but that theſe figures are equal. Axiom VIII. Now try the converſe. If two right lined figures are equal, then if they be laid the one on the other, their boundary lines will exactly coincide and agree. Who is there that does not ſee, that this propoſition (the converſe of the former) is by no means true? May not a triangle and a ſquare be equal figures; that is, have equal areas? And can a right lined figure, having only three ſides, laid upon another having four ſides, agree with it every where in the boundary lines? Again, if two triangles have their ſides reſpectively equal, their angles will alſo be reſpectively equal by the 8th. I. El. par.

39. But if two triangles have their angles respectively equal, it does not follow that their sides will be respectively equal; this may or may not be true, as it happens; as will be shown hereafter. Converse propositions therefore need a proof, notwithstanding this has been termed *superfluous* and *impertinent* by some who call themselves mathematicians *.

32. We will here remark the difference between a *definition* and a *proposition*. A definition, as we said, is an enumeration of all those ideas, which being united form a complex idea, to which we give a certain name. A definition then cannot be false; for it is giving a name to a collection of ideas: and if we give it that name; that is the name of it with us; that is the name we call it by; when we pronounce the name, we join that particular complex idea to that sound. A definition may be absurd or improper; we may chuse to give a name of our own to a complex idea, which all the world has agreed to call by another name; and perhaps all the world has agreed to fix a very different idea to our name, than what we affix to it; and thus we shall indeed be like the builders of Babel: much confusion will follow, and often does, but here is no agreement of ideas, affirmed or denied, and therefore nothing false. A definition may also be *obscure* in many ways, but it cannot be *false*. Euclid therefore does not DEMONSTRATE his DEFINITIONS; this would be absurd. He does not DEMONSTRATE his AXIOMS and POSTULATES; they need no demonstration; they are self-evident. He DEMONSTATES only his PROPOSITIONS, whether they be PROBLEMS or THEOREMS.

* See preface to *Emerson*'s Geometry, page VI.

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REMARKS ON PROPOSITIONS IN BOOK THE FIRST.

33. Prop. 2. and 3. These problems may seem to some very trifling. For who is there, who knows what a ruler and compasses are, but can *set off from a given point, a straight line of a given length?* But Euclid's design is not to teach the practical delineation of figures; but to derive the possibility of this in idea, from the postulates and axioms before laid down, as has been already observed. Whether we are able with the compasses to draw a circle that is truly and exactly round, is nothing to the purpose; in the ideal circle all the radii are exactly equal, the definition constitutes them equal; it supposes and declares they are so: therefore they are equal, and the circle is truly round in idea, whether or no it be round in the figure delineated on paper.

34. Prop. 4. It is worth while to remark with what caution and accuracy all Euclid's propositions are worded. A careless writer might say, If two triangles have two sides, and an angle equal, then their third side will also be equal. But Euclid cautions you not only that the sides must be equal, *each to each*; but also that the angle spoken of must be that which is *included* between the respectively equal sides. We will show that two triangles may have (as was said) two sides respectively equal, and also one angle, yet neither their third side nor the figures themselves be equal.

35. Let *ABC* (fig. 1.) be an isosceles triangle, *A* the vertex, *BC* the base. Produce the base to *D*, and join *AD*: then we shall have two triangles formed, viz. *ABD* and *ACD*, having two sides and an angle respectively equal; that is, the side *AB* in the triangle

gle *ABD*, equal to the ſide *AC* in the triangle *ACD*, alſo the ſide *AD* common to both triangles. The angle *ADC* is alſo common to both triangles; yet the third ſide *BD* in the former triangle, is not equal to the third ſide *CD* in the latter triangle: for *CD*, by the conſtruction, is only a part of *BD*. Nor are the figures *ABD* and *ACD* equal; for the former contains the latter.

36. PROP. 5. The angles on the other ſide of the baſe, are often called, the angles *under* the baſe.

COR. to PROP. 5. * Let *ABC* (fig. 2.) be an equilateral triangle; then becauſe *AB* and *AC* are equal, the triangle may alſo be conſidered as an iſoſceles triangle, and the other ſide *BC* as the baſe: therefore by this propoſition, the angles at the baſe *ABC* and *ACB* are equal. Again; becauſe the ſides *BA* and *BC* are equal, it may be conſidered as an iſoſceles triangle whoſe baſe is *AC*; therefore by this propoſition, the angles at the baſe *BAC* and *ACB* are equal: therefore becauſe *ABC* and *BAC* are both equal to *ACB*, they are alſo equal to each other, and thus all the three angles are equal, that is, the triangle is equiangular.

37. PROP. 6. In the demonſtration, after the words *the angle* DBC *is equal to the angle* ACB, add *by the ſuppoſition*.

The demonſtration here given, is that kind of argument the logicians call *Reductio ad abſurdum*, viz. proving the ſuppoſition of the contrary to what is affirmed in the propoſition, to be abſurd and impoſſible.

The corollary to this propoſition, is demonſtrated in the ſame manner as the corollary to propoſition the 5th.

* A COROLLARY is ſome truth obtained in conſequence of the demonſtration of a propoſition, over and above the propoſition itſelf. Such truths *for the moſt part* follow ſo evidently, as not to require a formal proof.

This

This proposition is the converse of proposition the 5th.

38. PROP. 7. The difficulty both in this and the former proposition lies in this; that we are to suppose an impossibility possible; and to delineate a figure to suit an impossible case. We cannot delineate an impossibility, we are therefore to draw a possible figure, and suppose it to be what it is not. Thus we are to suppose (in fig. 3.) not only the lines *CA* and *DA* to be equal, but also *CB* and *DB* to be equal. The figure may indeed be so drawn, that one pair of those lines, for instance *CA* and *DA*, may be actually equal, but not that the other pair also, viz. *CB* and *DB*, shall be equal, unless the points *C* and *D* coincide, and the two triangles *ACB* and *ADB* become one.

39. COR. to PROP. 8. Hence if two triangles have all their sides respectively equal, the triangles will be equal, and will have all their angles respectively equal: in other words, triangles equilateral to one another, are also equal, and equiangular to one another.

For the angle *BAC* * is equal to the angle *EDF* by this proposition; and the two sides *AB* and *AC* are equal to the two sides *DE* and *DF*, each to each, by the supposition; therefore the two triangles are equal, and the remaining angles to which equal sides are opposite, are equal by 4 I. El.

The converse of this corollary, viz. triangles equiangular to one another, are also equilateral to one another; is not true, as was before noted, par. 31.

40. COR. to PROP. 11. This corollary is not in Euclid. It is the foundation of the 11th axiom, and it is virtually supposed in the demonstration of prop. 4. I. El. where it is affirmed, that if *A* coincide with *D*, and *AB* with *DE*, the point *B* must coincide with *E*; which is not necessarily true, if two straight lines may have a common segment.

* Where no figure is mentioned, we refer to the figure in Dr. Simson's Euclid.

41. A.

41. A. PROP. 14. This is the converſe of propoſition 13.

Prop. 18 and 19, alſo 24 and 25 are converſe propoſitions.

41. B. PROP. 26. We may remark in this propoſition the ſame caution we before obſerved in par. 34. The two ſides (one in each triangle) which are thus ſuppoſed equal, muſt either lie between the angles in each triangle reſpectively equal, or be oppoſite to an equal angle in each triangle; otherwiſe the two triangles will not be equal *.

42. PROP. 29. This is the converſe of propoſitions 27 and 28. It is for the ſake of demonſtrating this propoſition, that Euclid aſſumes that axiom (the 12th) ſo much complained of by all learners: indeed it is ſo far from being ſelf-evident, that it has given the commentators much trouble to demonſtrate it.

If the axiom in par. 27. be admitted, the former part of this propoſition may be thus demonſtrated; ſee fig. 24.

For if *AGH* be not equal to *GHD*, one of them muſt be greater than the other; let *AGH* be the greater. At the point *G* make the angle *IGH* equal to the angle *GHD*, by 23. I. El.: then *IG* is parallel to *CD*, by 27. I. El. But, by the ſuppoſition, *AG* is alſo parallel to *CD*; therefore *AG* and *IG* meeting in one point *G*, are both parallel to *CD*, which is impoſſible by the axiom: therefore the angle *AGH* is not unequal to *GHD*, that is, it is equal to it.

The latter part of the demonſtration may proceed as in Simſon, beginning at the words, *but the angle* AGH, &c.

* As an inſtance of this, in the right angled triangle *BAC* (ſee the figure to prop. 8. VI. El.) let a perpendicular *AD* be drawn from the right angle at *A*, to the baſe *BC*; and it will divide the whole triangle into two others *ADB* and *ADC*, having two angles equal each to each, viz. *ADB*=*ADC* and *ABD*=*CAD*, (as is there ſhown) alſo the ſide *AD* is common to both triangles: yet theſe two triangles are not neceſſarily equal, becauſe the ſide *AD* neither lies between, nor is oppoſite to equal angles.

Co-

Corollaries to prop. 32. to come before that given by Dr. Simſon.

43. Cor. 1. The difference between the external angle, and either of the two internal and oppoſite angles, is equal to the other internal and oppoſite angle.

44. Cor. 2. If two angles of one triangle, are equal to two angles of another triangle, either ſingly each to each, or taken together, then the third or remaining angle in each triangle is equal.

45. Cor. 3. If one angle in any triangle, be either a right angle or an obtuſe angle, then each of the other two angles muſt be acute.

46. Cor. 4. If one angle in any triangle be a right angle, then the other two angles taken together make a right angle. Two ſuch angles are called *complements* of each other to a right angle. This may be expreſſed thus: In a right angled triangle, the acute angles are complements of each other.

47. Cor. 5. If the vertical angle of one iſoſceles triangle, be equal to the vertical angle of another iſoſceles triangle, then the angles at the baſe in one, are equal to the angles at the baſe in the other. Alſo, if one of the angles at the baſe in one iſoſceles triangle, is equal to one of the angles at the baſe in another, then the other angles are reſpectively equal.

48. Cor. 6. In an equilateral triangle, each angle is one third of two right angles; or two thirds of one right angle.

Cor. 7. For this read cor. 1. in Simſon.

49. Cor. 8. Hence all the interior angles of every rectilineal figure, are equal to twice as many right angles, except four, as the figure has ſides. For, let n be the number of ſides, s the ſum of all the interior angles, and let r ſtand for one right angle; then if cor. 1. in Simſon be algebraically expreſſed, it will ſtand thus, $s+4r=2\times r\times n$; whence $s=2\times r\times n-4r$.

50. Cor. 9. In every quadrilateral figure, the ſum of the four internal angles is equal to four right angles:

angles: for in this case $n=4$, therefore $s=2\times r\times 4 - 4r = 8r - 4r = 4r$, or four right angles.

51. This may be proved geometrically thus (from Mr. T. Simpson). Let *ABCD* (fig. 4.) be a quadrilateral figure. Draw *AC* through two opposite angles, and it will divide the figure into two triangles. The internal angles of each triangle are equal to two right angles, therefore the internal angles of both triangles are equal to four right angles: but the internal angles of both triangles make up the internal angles of the whole figure: therefore all the angles of a quadrilateral figure are equal to four right angles.

If to the quadrilateral figure, you add another side as *DE*, so that the whole figure *ABCDE* has now five sides, you then add another triangle *ADE*, and therefore add to the internal angles of the quadrilateral figure, two more right angles, making in the whole six right angles; and so on for as many sides as you please.

Cor. 10. For this read cor. 2. in Simson.

52. Prop. 34. After the definition of a parallelogram in prop. 34. we may add that of a *Trapezium*; by which is commonly meant *a four sided figure, of which the opposite sides are* not *parallel.* Furthermore: A line drawn through the opposite angles of a parallelogram is called a *Diameter.* A line drawn through the opposite angles of any quadrilateral figure (whether parallelogram or trapezium) is called a *Diagonal.*

53. Prop. 34. The converse of the first part of this proposition is this. If the opposite sides of a quadrilateral figure be equal, they will also be parallel; that is, the figure will be a parallelogram.

This being a very useful proposition may be thus demonstrated.

Let *ABCD* (fig. 5.) be a quadrilateral figure, whose opposite sides are equal. Draw the diagonal *BC*: then because *AB* is equal to *CD*, also *AC* equal to *BD*, and *BC* common to the two triangles *ABC* and

and *BCD*, the angles *ABC* and *BCD* are equal, by cor. to prop. 8. par. 39; but these are alternate angles, therefore *AB* and *CD* are parallel, by 27. I. El. Again; because *AB* and *CD* are equal and parallel, therefore *AC* and *BD* are parallel, by 33. I. El. But it was before proved, that *AB* and *CD* are parallel, therefore *ABCD* is a parallelogram.

Cor. Hence if the opposite sides of a quadrilateral figure be equal, the opposite angles will also be equal. For such a figure is a parallelogram, therefore the opposite angles equal by 34. I. El.

The converse of the second part of prop. 34, is this. If the opposite angles of a quadrilateral figure be equal, the opposite sides will be parallel; that is, the figure will be a parallelogram. For the proof see Dr. Simson's note on Def. 23. I. El.

54. Prop. 39. The converse of prop. 37, strictly speaking, is this. Equal triangles upon the same base, are between the same parallels. But this is not true, unless the triangles be also *on the same side of the base*; which is therefore added. The proof here given is that called *Reductio ad absurdum.*

55. Prop. 44. To apply a parallelogram to a straight line, is to make a parallelogram which shall have that line for one of its sides.

In prop. 42, we are taught to describe a parallelogram, that shall be equal to a given triangle, and have one of its angles equal to a given angle. We are here taught to describe a parallelogram having the two conditions before laid down, and also one more, viz. that one of its sides shall be equal to a given line.

56. Prop. 45. We are here taught how to describe a parallelogram equal to ANY rectilinear figure and having an angle equal to a given angle; but we are not here tied down to that other condition, viz. that one of the sides of this parallelogram shall be equal to a given right line.

The demonſtration is manifeſtly divided into two caſes; one where the rectilinear figure has four ſides only, the other where it has more than four. This laſt caſe is not ſpecified in Euclid, but it follows from the former.

In the former caſe then, where the rectilinear figure has four ſides only, you are to divide it by a diagonal *BD* into two triangles *ABD* and *DBC*. And, firſt, we are to make a parallelogram equal to the former triangle, having an angle at the baſe equal to the given angle by prop. 42. We are next to apply to the ſide of this parallelogram another or ſecond parallelogram, equal to the latter triangle, and having the angle at the baſe alſo equal to the given angle by prop. 44. Now Euclid ſhows that the baſes of theſe two parallelograms, which thus adhere together, form one continued right line. The ſame he ſhows of their tops, or the ſides oppoſite to their baſes; wherefore the two parallelograms together, form one great parallelogram. Now as the firſt parallelogram is equal to the firſt triangle, and the ſecond parallelogram to the ſecond triangle, both parallelograms together, or the great parallelogram *FKML*, is equal to both triangles, or to the quadrilateral *ABCD*, and has an angle *FKM* equal to the given angle.

57. Suppoſe now the original rectilinear figure had five ſides, viz. *AEBCD* (fig. 6.). We have already made a parallelogram *FKML* equal to the four ſided figure *ABCD*, and having the given angle. To the ſide *LM*, apply another parallelogram *LMNP*, equal to the triangle *AEB*, and having the angle *LMN* equal to the given angle. This parallelogram will alſo unite with the former, making one great parallelogram *FKNP*; as may be proved by the arguments before uſed. But the former parallelogram *FKLM* is equal to the trapezium *ABCD*, and the latter parallelogram *LMNP* is equal to the triangle *AEB*; wherefore both together or the great parallelogram *FKNP*, is equal to the five ſided figure *AEBCD*; and this

this parallelogram has alſo (as before) an angle equal to the given angle: and ſo on.

58. The firſt parallelogram was made equal to the firſt triangle by the 42d prop. which does not include this condition, that one of the ſides ſhall be equal to a given right line. But by the 44th, a parallelogram may be made which ſhall be equal to the firſt triangle, have an angle equal to the given angle, and alſo have the ſide *FK* equal to a given line, and of courſe have the oppoſite ſide *GH* equal to the given line by 34. I. El. As the ſecond parallelogram is applied to this line *GH*, its oppoſite ſide *LM* muſt be equal to *GH*, that is, to *FK*; and ſo the great parallelogram (be it compoſed of ever ſo many leſs ones) will always have the ſide *FK* (and its oppoſite) equal to the given line. And thus it appears how we may make a parallelogram, having one ſide *FK* equal to a given line, one angle *FKN* equal to a given angle, and the whole parallelogram *FKNP* equal to a given (five ſided) figure *AEBCD*: and ſo of any other rectilinear figure.

59. The text of Euclid (in prop. 45.) goes no further than the propoſition itſelf. The corollary is not Euclid's, but his tranſlator Commandine's. The latter part, beginning, "by applying," &c. is Dr. Simſon's: this being too conciſe to be underſtood by learners, we have here enlarged upon it.

60. Prop. 47. It ſhould be obſerved, that in a right angled triangle, the ſide ſubtending the right angle, is called the *hypothenuſe*. One of the other ſides is called the perpendicular, the other ſide the baſe, according to the poſition in which the triangle is drawn. Alſo both the ſides indifferently are called the legs of the right angled triangle.

The hypothenuſe may alſo be conſidered as the baſe of the triangle, if need require, as in prop. 8. VI. El.

61. A practical proof of the 47th of I. El. by cutting the two ſquares on the legs into ſuch pieces as being ſet together, form the ſquare on the hypothenuſe.

LEMMA. Let *ABCD* (fig. 7.) be a ſquare; from each angular point, take the equal diſtances *AE, BF, CG, DH* on each ſide reſpectively. Join the four points, *E, F, G, H.* The figure formed thereby ſhall be a ſquare.

From each ſide of the original ſquare take the equal parts *AE, BF, CG, DH,* and the remainders *EB, FC, GD, HA,* will be equal; and the angles at the four corners *ABCD* are equal as being right angles; therefore the four right angled triangles at the four corners of the original ſquare, are in all reſpects equal by 4. I. El. and the four ſides of the inner figure, viz. *EF, FG, GH, HE* are equal. Moreover, the angle *BEH* equals *EAH*+*AHE*, by 32. I. El. But *BEF* =*AHE*; ſubtract this latter equation from the former, and we have *FEH*=*EAH*= a right angle. In like manner may every angle of the inner figure *EF, GH* be ſhown to be a right angle; but its ſides are alſo equal, therefore it is a ſquare.

Otherwiſe thus: The four ſides of the inner figure being equal are therefore parallel, by converſe of 34. I. El. par. 53. And one angle *FEH* being a right angle (as before) all the angles are right angles, by cor. 46. I. El. therefore it is a ſquare.

62. This premiſed; through the point *F* draw *FN* parallel to *DC* (31. I. El.), therefore alſo parallel to *AB* (30. I. El.) Alſo draw *GM* and *EL* parallel to *BC*, and therefore parallel to *AD*. Then firſt of all *FCGM* is a parallelogram, and *FG* the diameter; therefore the triangle *FMG* is every way equal to *GCF*; that is, every way equal to *EAH*. In like manner the triangle *ELF* is every way equal to the triangle *FBE*; that is, every way equal to *HDG*. Again, *AELN* is a ſquare upon *AE*, the greater leg of one of theſe equal right angled triangles. *MGDN* is a ſquare upon *GM*, the leſs leg of one of theſe equal right angled triangles, and *EFGH* is a ſquare upon the hypothenuſe. From the former of theſe ſquares cut off the corner *EAH*, and it will fit the triangle

 FMG.

FMG. Cut off the corner *HDG* (part of the two ſquares on the legs) and it will fit the triangle *ELF*. Thus the ſquare *EFGH* will be wholly filled up with parts of the two ſquares *AELN* and *MGDN*, and no parts of thoſe two ſquares remain. Therefore the former ſquare is equal to both the latter ſquares taken together; which was to be ſhown.

63. Many of the propoſitions in the firſt book of Euclid, are of no other uſe than as mediums to prove others. Thus the 16th is manifeſtly implied in the 32d, and therefore uſeleſs after the 32d is demonſtrated. The propoſitions are alſo ranged, not in the order of their ſubjects, but in ſuch an order as the argument requires. We have here endeavoured to range them according to the nature of the ſubject; omitting ſuch as are of little uſe in themſelves.

64. Propoſitions relating to RIGHT LINES.

1 An equilateral triangle may be made upon a given right line. *Prop.* 1.

2 A given right line may be biſected, or divided into two equal parts geometrically. *Prop.* 10.

3 A right line may be drawn at right angles to another right line from a point therein.——This is called *erecting a perpendicular*. *Prop.* 11.

4 A right line may be drawn at right angles to another right line, from a given point without it.—This is called *letting fall a perpendicular*. *Prop.* 12.

65. Of ANGLES.

5 An angle may be biſected geometrically. *Prop.* 9.

Definition. If one right line croſſes another, there will be four angles formed. Thoſe two angles which are on the ſame ſide of one of the lines, ſo as to be contiguous, are called *adjacent* angles: thoſe two angles which are on contrary ſides, ſo that only their vertexes meet, are called *vertical* or *oppoſite* angles. The two lines which form an angle are ſometimes called its *legs*.

6 The two adjacent angles taken together are equal to two right angles. *Prop.* 13. And converſely, if two

adjacent angles are equal to two right angles, the extreme legs of theſe angles form one right line. *Prop.* 14.

7 Vertical or oppoſite angles are equal. *Prop.* 15.

66. Properties of PARALLEL LINES.

Definition. If a line falls on two parallel lines, it makes with each of them four angles, in all eight. The four angles within (or between) the parallel lines are called *internal* angles; the other four without the parallel lines, are called *external* angles.

Theſe angles are alſo conſidered with reference to the cutting line; they are either on the ſame ſide that line, or on oppoſite ſides.

The interior angles on oppoſite ſides of the cutting line, are called *alternate* angles.

The angles formed at one parallel line, are ſaid to be oppoſite angles to thoſe formed at the other parallel line.

8 If a right line falling upon two others, makes the alternate angles equal; then thoſe two lines are parallel. *Prop.* 27.

9 If the exterior angle be equal to the interior and oppoſite angle, on the ſame ſide of the cutting line; then thoſe two lines are parallel. *Prop.* 28.

10 If the two interior angles on the ſame ſide the cutting line, taken together, be equal to two right angles; then thoſe two lines are parallel. *Prop.* 28.

11 The converſe of theſe three propoſitions is proved in *Prop.* 29.

12 Right lines which are parallel to the ſame right line, are parallel to one another. *Prop.* 30.

67. Of TRIANGLES COMPARED with one another.

There are three caſes in which two triangles are in all reſpects equal.

13 *Firſt,* When all three ſides are reſpectively equal (that is, each to each), then the angles oppoſite to equal ſides are alſo equal. *Cor.* to *Prop.* 8. *par.* 39.

14 *Secondly,* When two ſides, and the angle contained by thoſe ſides are reſpectively equal, then their third ſides are equal; and the angles oppoſite to equal ſides are alſo equal. *Prop.* 4.

15 *Thirdly,* When two angles and one ſide are reſpectively equal; and that ſide either lies between the two angles, or is oppoſite to one of the equal angles,

angles, then their third angle is alſo equal; and the other ſides, oppoſite to equal angles, are equal. *Prop.* 26.

68. Properties of all TRIANGLES in GENERAL.

16 In every triangle the greater ſide of any two, is oppoſite to, or ſubtends the greater angle. *Prop.* 18. And converſely, the greater angle is ſubtended by the greater ſide. *Prop.* 19.

17 Any two ſides of a triangle taken together, are greater than the third ſide. *Prop.* 20.

18 If one ſide of any triangle be produced, the external angle is equal to the two internal and oppoſite angles. And the three interior angles of every triangle, are equal to two right angles. *Prop.* 32. and *Cor.*

19 Triangles upon the ſame baſe, or upon equal baſes, and between the ſame parallels, are equal. *Prop.* 37. and 38. And converſely; equal triangles, on the ſame baſe, and on the ſame ſide; or upon equal baſes in the ſame ſtraight line, and towards the ſame parts; are between the ſame parallels. *Prop.* 39. and 40.

69. Properties of PARTICULAR TRIANGLES.

20 In an ISOSCELES triangle, the angles at the baſe are equal to each other. And if the equal ſides be produced, the angles under the baſe are alſo equal to each other. *Prop.* 5. And converſely, if two angles of a triangle are equal to one another, the ſides which ſubtend the equal angles are equal; that is, the triangle is iſoſceles. *Prop.* 6.

Definition. In a right angled triangle, the ſide which ſubtends the right angle, is called the *hypothenuſe.*

21 In a RIGHT ANGLED triangle, the ſquare on the hypothenuſe is equal to the ſum of the ſquares on the two ſides which contain the right angle. *Prop.* 47. And converſely, if the ſquare upon one ſide a triangle be equal to the ſum of the ſquares upon the other two ſides, the angle contained by thoſe two ſides is a right angle. *Prop.* 48.

70. Properties of PARALLELOGRAMS.

22 Right lines which join the extremities of equal and parallel lines, are alſo parallel. That is, theſe four lines make a parallelogram. *Prop.* 33.

23 Oppofite fides and angles of a parallelogram are equal to one another, and the diameter bifects the parallelogram. *Prop.* 34. The converfe is proved, *par.* 53.

24 If one angle of a parallelogram be a right angle, all the angles are likewife right angles. *Cor.* to *Prop.* 46.

25 Parallelograms on the fame, or on equal bafes, and between the fame parallels, are equal to one another. *Prop.* 35. and 36.

26 If a parallelogram and a triangle be upon the fame bafe and between the fame parallels, the parallelogram fhall be double of the triangle. *Prop.* 41.

27 The complements of the parallelograms which are about the diameter of any parallelogram, are equal to one another. *Prop.* 43.

71. General properties of all RIGHT LINED FIGURES.

28 All the internal angles of any right lined figure taken together, are equal to twice as many right angles, except four, as the figure has fides. *Cor.* 8. *Prop.* 32. *par.* 49—51.

29 All the external angles of any right lined figure, made by each fide produced the fame way, taken together, are equal to four right angles. *Cor.* 2. *Prop.* 32. in Simfon.

We may confider the above, as an abftract of the firft book of Euclid.

BOOK THE SECOND.

72. All lines are meafured by other lines. Some one line, arbitrarily affumed, is called unity, and the length of every other line is reprefented by the number of thofe lines, called units contained in it. Thus if a line an inch long be called unity, then the length of any other line as a cubit, is meafured by the number of inches it contains. If the cubit contains 21 fuch lines (each an inch long) then the cubit is 21 inches long. Or rather, the proportion of the line called

called a cubit, to the line called an inch, is that of the number 21, to the number 1. And thus lines are compared with lines, but cannot be compared with surfaces, or with solids; any more than an inch can be compared with an hour.

73. In like manner, surfaces are measured by other surfaces. If a right angled parallelogram, or rectangle, has its length and breadth marked out or divided into inches, and through each point of division, lines be drawn parallel to the sides of the rectangle, the figure by this means will be marked out into squares, each square being one inch in the side; and the surface of the whole figure will be measured by the number of these squares (called square inches or superficial inches) it contains. Now the number of these square inches will be found by multiplying the number of linear inches contained in one side of the rectangle, by the number of linear inches contained in the other. For there will be as many square inches in each row of squares, as there are linear inches in one side of the rectangle, and as many such rows of squares, as there are linear inches in the other side, as appears by the inspection of the figure. Now suppose the length of such a rectangle is 9 inches, and the breadth 6 inches, then the whole surface contains 54 square inches: that is, the whole surface of this rectangle, is to the surface of a square, whose side is one inch, in the proportion of the number 54 to the number 1. And thus surfaces are compared with surfaces, but not with lines or solids.

74. Hence it follows, that such rectangles will be represented by the product of the multiplication of the number of lines called units in one side, by the number of lines called units in the other side. I say *represented*, that is, whatever proportion any two rectangles have to each other, the same will be the proportion of the two products of the multiplication of the number of lines called units, in their sides respectively. Hence rectangles in geometry, and products in arithmetic, are put for each other, and the

names

names applied promiſcuouſly. Hence, what is ſhown concerning the equality of certain rectangles in the ſecond book of the Elements, will alſo be true of the product of the multiplication of the number of lines called units in their ſides.

75. Hence if the number of theſe lines called units in the ſide of a rectangle, or in the parts into which the ſide of the rectangle is divided, be repreſented algebraically by letters; moſt of the propoſitions in this ſecond book may be demonſtrated by the rules of algebra, as well as geometrically; as ſhall be ſhown.

76. Prop. 1. Let the whole line BC be called s, and let the ſeveral parts BD, DE, EC, be called a and b and c; then is $s=a+b+c$. Call the other undivided line x. Multiply this equation by x, and we have $sx=ax+bx+cx$, that is, the rectangle sx contained by the intire lines s and x, is equal to the ſeveral rectangles ax, bx and cx contained by the undivided line x, and the ſeveral parts a and b and c of the divided line s.

77. Prop. 2. Let s be the whole line, a and b the parts; then we have $a+b=s$. Multiply this equation by s, and we have $as+bs=ss$, that is, the rectangles as and bs contained by the whole and each of the parts are equal to ss, the ſquare of the whole.

78. Prop. 3. Let the whole line be s, the two parts a and b; then we have $s=a+b$. Multiply this equation by b, and we have $sb=ab+bb$, that is, the rectangle sb, contained by the whole of s and one of the parts b, is equal to the rectangle ab contained by the two parts a and b together with bb, the ſquare of the aforeſaid part b.

79. Prop. 4. Let the whole line be s, the two parts a and b, as before; then is $s=a+b$. Square both ſides of the equation, and we have $ss=aa+bb+2ab$; that is, ss, or the ſquare of the whole line, is equal to $aa+bb$, the ſquares of the two parts, together with $2ab$, or twice the rectangle contained by the parts.

80. In the demonſtration, when the figure $CGKB$

is

is to be proved rectangular, Euclid instead of referring to Cor. 46. I. El. repeats the demonstration of that proposition. Therefore after the words "it is likewise rectangular," read thus, "For *CBK* is a right angle, therefore *CGKB* is rectangular by Cor. 46. I. El." Then go on at the words "but it is also equilateral," &c.

81. Prop. 5. The best way to demonstrate this is to put a for each of the equal parts *AC* and *CB*; and x for the line *CD* between the points of section; then the greater of the unequal parts, or *AD*, is $AC+CD$ or $a+x$, and the less of the unequal parts or *DB*, is $CB-CD$ or $a-x$; their rectangle or $\overline{a+x} \times \overline{a-x} = aa-xx$: to this add the square of the line between the points of section or xx, and the sum is aa, or the square of one of the equal parts, or the square of half the line *AB*.

82. Prop. 6. Here again, it is best to put a for one of the equal parts, consequently $2a$ for the whole line. Let the part produced be x, then the whole line thus produced is $2a+x$, and the rectangle under that line and the part produced is $\overline{2a+x} \times x = 2ax + xx$; to this add the square of half the line bisected or the square of a, or aa, and the whole is $aa+2ax+xx$, which is equal to the square of $a+x$, or the line made up of the half a, and the part produced x, as will appear by multiplying $\overline{a+x}$ by $\overline{a+x}$ algebraically.

83. Prop. 7. Let the whole line be s, and the two parts a and b, as before; then $s=\overline{a+b}$, and $ss = aa+2ab+bb$, or $2ab+bb+aa$. To both sides add bb, and we have $ss+bb=2ab+2bb+aa$; but $2ab+2bb=2\times\overline{a+b}\times b=2sb$, therefore $ss+bb=2sb+aa$; that is, the square of the whole line (or ss), and the square of one of the parts (or bb), is equal to twice the rectangle contained by the whole and that part (or $2sb$), together with (aa) the square of the other part.

84. Prop. 8. Let the whole line be s, the two parts a and b, as before; then $a+b=s$, and $a=s-b$ and

and $aa = ss - 2sb + bb$. To both sides add $4sb$, and we have $4sb + aa = ss + 2sb + bb = \overline{s+b}|^2$, by prop. 4; that is, four times sb, the rectangle contained by the whole line and one of the parts b, together with the square of the other part a, is equal to the square of $s+b$, the straight line which is made up of the whole and that part.

85. Prop. 9. Let each of the equal parts be a, and the line between the points of section x, as before; then is the greater of the unequal parts $a+x$, and the less $a-x$, as in prop. 5. The square of the former is $aa + 2ax + xx$, the square of the latter $aa - 2ax + xx$. Add these together, and their sum is $2aa + 2xx$, or $2 \times \overline{aa+xx}$; that is, the sum of the squares of the two unequal parts, is double the sum of the squares of half the line, and of the part between the points of section.

86. Prop. 10. Let the two equal parts be a, and the additional part d: then is the whole line and the part produced $2a+d$, and half the line and the part produced $a+d$. The square of the former is $4aa + 4ad + dd$: to this add dd, and the whole is $4aa + 4ad + 2dd$. Again, the square of half the line and the part produced, or the square of $a+d$, is $aa + 2ad + dd$: to this add aa, and the whole is $2aa + 2ad + dd$; but the former sum $4aa + 4ad + 2dd = 2 \times \overline{2aa + 2ad + dd}$, or twice the latter: that is, the square of the whole line when produced, and the square of the additional part, is double the square of half the line or a, and the square of $\overline{a+d}$ or half the line and the additional part.

87. Prop. 11. This problem is again proposed in another form in prop. 30. book 6th. In these notes we have shown the reason of this difference, and also given a geometrical construction of the problem as proposed in the other form. It will be proper here to give the algebraic solution of the problem in its present form, and to show its connection with Euclid's geometrical construction given in this place.

88. To

88. To avoid fractions, it will be best to call AB (the given line) $2a$; that is, to put a for half that given line. Call the part to be squared, or AH, x, and the other part HB, y: then the two fundamental equations are $x+y=2a$ and $xx=2ay$. From the former of these we have $y=2a-x$ and $2ay=4aa-2ax$: substitute this for $2ay$ in the latter fundamental equation, and we have $xx=4aa-2ax$, whence $xx+2ax=4aa$ and $xx+2ax+aa=4aa+aa=5aa$; therefore $x+a=\pm\sqrt{5aa}$ and $x=\sqrt{5aa}-a$. We had $y=2a-x$: for x put its value now found, and $y=2a-\overline{\sqrt{5aa}-a}=3a-\sqrt{5aa}$.

89. Euclid's construction affords the very same algebraic value of the two parts x and y. For (by 47. I. El.) $AB^2+AE^2=EB^2$; but $AB=2a$, and $AE=a$, therefore $AB^2=4aa$, and $AE^2=aa$, and $EB^2=5aa$, and $EB=\sqrt{5aa}$; but AH or $x=AF=EF-EA$, or $EB-EA=\sqrt{5aa}-a$, as before.

Again: Produce AF to L, so that AL may be equal to AB; then is $FL=AL-AF=AB-AH=HB$; but FL is also equal to $EL-EF$, therefore HB or $y=EL-EF$ or $EL-EB$ or $3a-\sqrt{5aa}$, as before.

As to the other answer to this quadratic equation (viz. $x=-\sqrt{5aa}-a$, and $y=3a+\sqrt{5aa}$) it is evident, that here x is always negative, and y affirmative; therefore the first fundamental equation becomes $-x+y=2a$, but the other is the same as before. The problem therefore is changed into another, viz. To find two lines, having a given difference, and such that the rectangle under that difference and one of them, shall be equal to the square of the other. A problem we have no concern with.

90. Prop. 13. The demonstration of the second case of this proposition is very difficult to learners. It may be proved in the same manner as the first, making a slight alteration in the beginning. Thus: Secondly, Let AD fall without the triangle ABC, and

because

because the straight line *BD* is divided into two parts in *C*, the squares of *CB* and *BD*, &c. as in the first case.

BOOK THE THIRD.

91. Def. 7. This definition is necessary to understand the latter part of prop. 31. Dr. Simson has left out that part in the enunciation of prop. 31. yet explains it at the end of the demonstration. Both this definition, and that latter part of prop. 31. may be omitted.

92. Prop. 2. It will conduce to the clearer understanding of Dr. Simson's demonstration, if it be observed that the points *A* and *B* are in the circumference of the circle; and we are to show the absurdity of supposing the line *AB*, which joins them, to fall without the circle. We are therefore obliged to make a false representation of the circle, and to suppose its circumference not to be round; we are obliged to suppose the circumference of the circle to be the line *CAFBC*, so that the right line *AEB* joining the points *A* and *B*, may fall wholly without *AFB*, the part of the circumference intercepted between the same points *A* and *B*. We have before noted the difficulty that may be expected from trying to draw a figure to suit an impossible case. The line *CAFBC* not being in appearance round, is an aukward representation of a circle: yet *suppose* this, so that the right line *AEB* may fall without *AFB*, a part of that circle, and it will imply that *DF* is less than *DE*: but if *AFB* is a part of a circle, then *DF* is a radius, and equal to *DB*, and therefore greater than *DE*. That is, *DF*, which by the supposition (that the line *AB* falls without the circle) is less than *DE*, must (from the nature of the circle) be also greater than *DE*; which is absurd: therefore the supposition is impossible.

The demonstration in Simson is Euclid's own; and is a *reductio ad absurdum*. Commandine gives another demon-

demonſtration which is a direct proof. As a direct proof is more ſatisfactory than an indirect one, we will give Commandine's; but it requires, that we admit an axiom not laid down by Euclid, and which indeed is the reaſon why Euclid prefers the indirect proof.

AXIOM. If a point be taken nearer to the center of a circle than the circumference, that point falls within the circle. *

This admitted; let *ACB* (fig. 8.) be a circle, *D* the center, *A* and *B* two points in the circumference. Join *AB*: I ſay the line *AB* falls wholly within the circle. For take any point *E* in that line, and join *DA*, *DE* and *DB*: becauſe *DA* is equal to *DB*, the angle *DAB* is equal to the angle *DBA* (5. I. El.). But *DEB* is greater than *DAB* (16. I. El.), and therefore greater than its equal *DBA*, or *DBE*; therefore the ſide *DB* is greater than the ſide *DE* (by 19. I. El.), or which is the ſame, *DE* is leſs than *DB*, and the point *E* is nearer to the center than the point *B*: but the point *B* is in the circumference, therefore the point *E* is nearer the center than the circumference, and by the axiom falls within the circle. The ſame may be ſhown of every other point in the line *AB*, therefore that line falls wholly within the circle.

93. PROP. 16. The proof given by Euclid is what is called *reductio ad abſurdum*. Let us admit an axiom correſponding to what we had in the laſt paragraph, and we ſhall have a direct proof analogous to the former.

AXIOM. If a point be taken further from the center of a circle, than the circumference; that point falls without the circle.

Let *ABE* (fig. 9.) be a circle, *D* the center, *AB* a diameter: through *A* draw *TAC* at right angles to *AB*: I ſay the line *TAC* touches the circle in *A*.

Take any point *C* in the line *TAC*, and draw *DEC* cutting the circle in *E*; then becauſe the triangle *DAC* is right angled at *A*, the other two angles are

each leſs than a right angle (32. I. El. cor. 3. par. 45.) therefore the ſide *DC* is greater than the ſide *DA* (19. I. El.), and therefore greater than its equal *DE*. The point *C* therefore is further from the center *D* than the point *E*, which is in the circumference; therefore by the axiom, the point *C* falls without the circle. The ſame may be ſhown of every other point in the line *TAC*, except *A*, which by the ſuppoſition is in the circumference. The line *TAC* therefore touches the circle in the point *A*, but does not cut it.

94. PROP. 31. The latter part of this propoſition as it ſtands in the Greek, and in Commandine's tranſlation, is left out in Dr. Simſon's tranſlation. The truth is, this latter part is not only of little uſe, but alſo obſcure, and therefore may be omitted.

Though Dr. Simſon has not put this latter part into the enunciation, yet he explains it at the end of the demonſtration. The expreſſions in the enunciation, and in this latter part, (beginning thus, "*Beſides it is manifeſt*,") ſeem to contradict one another, but it is not ſo. In the former; the angle *in* the greater ſegment is ſaid to be leſs than a right angle. In the latter; the angle *of* the greater ſegment is ſaid to be greater than a right angle. So again, in the former; the angle *in* the leſſer ſegment is ſaid to be greater than a right angle: in the latter; the angle *of* the leſs ſegment is ſaid to be leſs than a right angle. See Def. 7.

The latter part may be thus explained.

There are two right angles *CAB* and *CAF*, alſo two ſegments of the circle; the greater ſegment *ABC*, and the leſs ſegment *ADC*. Now in the former of theſe, the ſegment *ABC* includes the right angle *CAB*; but in the latter, the right angle *CAF* includes the ſegment *ADC*. Therefore the angle *of* the former, or greater ſegment, is ſaid to be greater than the right angle; and the angle *of* the latter, or leſs ſegment, is ſaid to be leſs than a right angle.

BOOK

BOOK THE FOURTH.

Inſtead of prop. 5. ſubſtitute the following prop. Or rather read both; this firſt.

95. Prop. 5. To deſcribe a circle that ſhall paſs through two given points *A* and *B* (fig. 18.)

Join the points *A* and *B*, biſect the line *AB* by the perpendicular *DF* (10. I. El.) and any circle whoſe center is ſome point in the indefinite line *DF*, for inſtance *F*, and whoſe radius is *FA*, will be that required.

For join *FA* and *FB*, and in the triangles *ADF* and *BDF*, the ſide *AD* in one is equal to the ſide *BD* in the other; the ſide *DF* is common, and the angle between theſe ſides is equal, therefore the third ſide *AF* in one is equal to the third ſide *BF* in the other (4. I. El.), and a circle deſcribed with the center *F*, and at the diſtance *FA* of one of the points, will paſs through the other.

96. Cor. 1. Hence we may learn to draw a circle that ſhall paſs through three given points *B*, *A* and *C*: for join the points *A* and *B*, biſect the line *AB* perpendicularly in *D*, and the center of every circle that paſſes through *A* and *B*, muſt be ſomewhere in the line *DF*. In like manner, join *A* and *C*, and biſect *AC* perpendicularly in *E*, and the center of every circle that paſſes through *A* and *C*, muſt be ſomewhere in the line *EF*. Therefore the center of the circle that paſſes through all three points *B*, *A* and *C*, muſt be a point, both in the line *DF* and alſo in the line *EF*; but ſuch a point can be no where but in the common interſection of thoſe lines *F*; which is therefore the center of the circle required.

Scholium. The line *DF* is called the locus of the centers of all the circles that will paſs through *A* and *B*. And the line *EF* is the locus of the centers of all the circles that will paſs through *A* and *C*. And this method of ſolving geometrical problems,

 by

by finding the *locus* of all those points, that will answer the several conditions separately, is called, *Constructing of problems by the intersection of* GEOMETRIC LOCI.

98. COR. 2. If the points *A*, *B*, and *C*, lie all in one right line, the perpendiculars *DF* and *EF* will be parallel, and never intersect; therefore in this case the problem is impossible.

99. COR. 3. Hence we may see why it is always possible (excepting in one case) to draw a circle to pass through any three points, but not through any four points. For two right lines not parallel (if continued) must have one common intersection somewhere. But in the case of four given points, there will be three perpendiculars; but three right lines have not of necessity one common intersection. They may or they may not intersect in one common point; therefore the problem to draw a circle through four given points, sometimes does and sometimes does not admit of a solution.

100. COR. 4. Hence we may learn to describe a circle about a given triangle. For nothing more is required, than to draw a circle that shall pass through the three angular points.

BOOK THE FIFTH.

101. The great difficulty in this book arises from hence, that Euclid extends his propositions about proportionals, to incommensurable as well as commensurable quantities.

102. And here in the first place we must explain what is meant by *commensurable* and *incommensurable* quantities. Now a less quantity is said to *measure* a greater, when the less is contained in the greater any exact, or integral number of times. Two quantities are said to be *commensurable* when a third quantity can be found that will measure them both; and that third quan-

quantity is called their *common meaſure.* Two quantities are ſaid to be *incommenſurable* when they admit of no common meaſure.

103. All whole numbers are commenſurable; for unity is their common meaſure. All fractions are commenſurable: for let any number of fractions be reduced to other equivalent fractions, having one common denominator; then a fraction whoſe numerator is unity, and denominator that common denominator, will meaſure every one of thoſe fractions.

104. *Example.* Let the fractions be $\frac{2}{3}$, $\frac{4}{5}$, $\frac{6}{7}$, Theſe reduced to others of the ſame value having one common denominator will be $\frac{70}{105}$, $\frac{84}{105}$, and $\frac{90}{105}$. Now it is evident, that the fraction $\frac{1}{105}$ will meaſure every one of them, becauſe the $\frac{1}{105}$th part of an unit will meaſure any integral number of the like $\frac{1}{105}$th parts of an unit.

All mixt numbers are alſo commenſurable; for they may be reduced to improper fractions: therefore numbers of all ſorts, integral, fractional or mixt, are commenſurable.

105. Two quantities having thus a common meaſure, can be repreſented by numbers, or are to each other, as one number to another; namely, as the *number* of times that common meaſure is contained in one quantity, is to the *number* of times that common meaſure is contained in the other quantity; juſt for the ſame reaſon as a line of 12 inches long, and a line of 5 inches long (having a line of one inch for their common meaſure) may be repreſented by the numbers 12 and 5; or are to each other as the numbers 12 and 5.

106. On the contrary, incommenſurable quantities,

admitting of no common meaſure, are not as number to number, nor can they be repreſented by any numbers whatever. We will give one inſtance of this out of many. The ſide of a ſquare is incommenſurable to the diagonal of that ſquare. If the ſide of the ſquare be 1, the diagonal will be more than 1, but leſs than 2. If the ſide be 10, that is, if the ſide be divided into 10 equal parts, the diagonal will be more than 14 ſuch parts, but leſs than 15; ſhould we ſay it was exactly 14 ſuch parts, the error would be leſs than one of theſe parts of which 10 make the ſide; that is, leſs than $\frac{1}{10}$th of the ſide. Again, let the ſide be divided into 100 equal parts, the diagonal will be more than 141 of ſuch parts, but leſs than 142. The error (ſhould we call the diagonal 141 of theſe parts) would be leſs than $\frac{1}{100}$th part of the ſide. The error thus made (by aſſigning an exact number of equal parts to both the ſide and the diagonal) will grow leſs and leſs, as the number of parts increaſes; yet that error can never be exhauſted, nor can that error be aſſigned in numbers. Though the number of equal parts into which the ſide is divided be ever ſo great; yet the diagonal will never contain any number of ſuch parts EXACTLY: and thus the ſide of the ſquare is NOT to the diagonal as number to number, or, it is INCOMMENSURABLE to the diagonal. That the ſide of the ſquare and its diagonal are thus related, is demonſtrated in the laſt propoſition of the tenth book of Euclid.

Having thus endeavoured to explain what is meant by commenſurable and incommenſurable quantities, we proceed to the definitions.

107. DEF. 3. The idea of ratio is undoubtedly a ſimple idea; therefore no definition of it can be given, and all controverſies about its definition are futile. The former of the two magnitudes, whoſe relation is here

here contemplated, is called the ANTECEDENT, the latter the CONSEQUENT. We may further obſerve, that the idea of ratio, reſults from comparing two things in reſpect of magnitude or quantity only. Quality (conſidered as addititious or ſubductitious) hath no concern with it.

108. DEF. 4. The meaning of this definition is, that all *homogeneous* quantities have a certain ratio to each other; but ratio cannot ſubſiſt between *heterogeneous* quantities. When the leſs of the two quantities thus compared, can be ſo multiplied as to exceed the greater, then thoſe two are *homogeneous* quantities. Thus a line of one yard, can be ſo multiplied as to exceed a mile in length; but it cannot be ſo multiplied as to exceed a year in duration; therefore a yard hath a certain ratio to a mile, but it hath none to a year. For a like reaſon, lines can have no ratios to ſurfaces, nor ſurfaces to ſolids; but lines are to be compared with lines, ſurfaces with ſurfaces and ſolids with ſolids.

109. Although the idea of proportionality is a ſimple idea, and cannot be conveyed by any words whatever; yet there may be a characteriſtic or mark which always accompanies it, and may direct us to know equality of ratios. The characteriſtic commonly given, viz. Def. 20 VIII. El. (which ſee, par. 124.) is true; and with reſpect to numbers and all commenſurable quantities, is general. But a mark for equality of ratios is wanted, which extends to incommenſurables alſo; ſuch is Euclid's 5th definition in this book. But to underſtand it, inſtances of its application in particular caſes muſt be given.

110. Suppoſe then the four numbers following were propoſed to our conſideration, viz. 3, 2, 12, 8, I ſay, the firſt of theſe has the ſame ratio to the ſecond, which the third has to the fourth; according to the teſt laid down in definition 5th: or, in the language of definition 6th, I ſay theſe four magnitudes are *proportionals*.

111. For, let us multiply the firſt and third by any number as 6, and they become $3 \times 6 = 18$ and $12 \times 6 = 72$. Again, multiply the ſecond and fourth by any number, as 10, and they become $2 \times 10 = 20$ and $8 \times 10 = 80$. Now as the multiple of the firſt, or 18, is *leſs* than the multiple of the ſecond, or 20, ſo is the equimultiple of the third, or 72, *leſs* than the equimultiple of the fourth, or 80.

112. Let us take the ſame equimultiples of the firſt and third, as before; but let us multiply the ſecond and fourth by the number 9, and they become 18 and 72. Here the multiple of the firſt is *equal* to the multiple of the ſecond, both being 18; and ſo the multiple of the third is *equal* to the multiple of the fourth, both being 72.

113. Let us take the ſame equimultiples of the firſt and third, as before; but let us now multiply the ſecond and fourth by the number 8, and they become 16 and 64. Here the multiple of the firſt, or 18, is *greater* than the multiple of the ſecond, or 16, and the multiple of the third, or 72, is alſo *greater* than the multiple of the fourth, which is 64.

114. The criterion is in this inſtance complete: it holds in all three circumſtances: whether the multiple of the firſt is leſs, equal to, or greater than the multiple of the ſecond; ſtill the multiple of the third is in like manner leſs, equal to, or greater than the multiple of the fourth. The criterion muſt hold in all three caſes; it muſt alſo hold *whatever* be the multiplier, to be complete.

115. It may be objected to this, that if it be thus neceſſary to ſhow that the criterion holds, whatever be the equimultiples of the firſt and third terms; and alſo whatever be the equimultiples of the ſecond and fourth; we ſhall never be able in this manner to determine whether four quantities are proportional; becauſe we can never make an actual trial of all poſſible multipliers. It may be anſwered, we are not to determine this queſtion by making ſuch trials;
but

but by ſhowing from the nature of the quantities, that whatever be the multipliers, if the multiple of the firſt exceeds that of the ſecond, the equimultiple of the third *will* exceed that of the fourth; and ſo on. This is Euclid's method in the very firſt inſtance of applying this definition, Prop. 1. VI. El. And in the ſame manner it is ſhewn in par. 129, that if $\frac{a}{b}=\frac{c}{d}$, then a, b, c, d, are proportional according to Euclid's 5th definition.

116. Suppoſe now A (fig. 10.) to be the ſide of a little ſquare, and B its diagonal. Suppoſe alſo C to be the ſide of a greater ſquare, and D its diagonal. Then if we take A 15 times, it will ſomewhat exceed the diagonal B taken 10 times. Alſo, if we take the ſide C 15 times, it will ſomewhat exceed the diagonal D taken 10 times. Again, if we take the ſide A 14 times, it will ſomewhat fall ſhort of the diagonal B taken 10 times: ſo alſo if we take the ſide C 14 times, it will ſomewhat fall ſhort of the diagonal D taken 10 times. Here it may be obſerved, that in the firſt caſe the equimultiplier of the firſt and third terms, viz. A and C is 15, the equimultiplier of the ſecond and fourth terms, viz. B and D is 10. In the ſecond caſe, the former equimultiplier is 14, the latter equimultiplier is 10. Now if this be true whatever be the multipliers; that is, if whenever the multiple of the firſt exceeds or falls ſhort of the multiple of the ſecond, the equimultiple of the third, in like manner, exceeds or falls ſhort of the equimultiple of the fourth; then are thoſe quantities proportional: that is, A is to B, as C is to D. And this holds though no multiplier can be found, by which multiplying the ſide A, it ſhall exactly equal the diagonal B. Thus then we have a criterion of the equality of two ratios, although the terms of thoſe ratios are incommenſurable to each other, and conſequently not expreſſible by numbers. It is ſufficient that whenever the multiple of the firſt term is leſs (*equal to*, is here ſuppoſed impoſſible)

possible) or greater than the second; the multiple of the third is likewise less or greater than the fourth.

117. Def. 6. This is merely an explanation of the word *proportional*. It may be observed, that proportion or proportionality, implies four terms. Ratio subsists between two terms only. Indeed ratio necessarily implies two terms, but an equality of ratios implies four terms; the two first of which are to be compared together, and do bear a ratio to each other (see Def. 4.); and the two last are also to be compared together, and do bear the *same* ratio to each other.

118. Def. 7. In the criterion of the equality of ratios given in definition 5, the multipliers may be any number whatever; but not so in the criterion here laid down of one ratio's being greater than another. The sense of this definition is, that if any two multipliers *can possibly be found* such, that "when equimultiples, &c." (as in definition 7th) that then the ratio of the first term to the second, is to be *reckoned* and *called* greater, than the ratio of the third term to the fourth, &c.

119. A question arises, How shall we know whether any such multipliers *can* be found? The answer is, Make two fractions, the first having for its numerator the antecedent, and for its denominator the consequent of the first ratio: let a second fraction be formed, in like manner, out of the terms of the latter ratio; then the numerator and denominator of either of these two fractions, or of any other fraction of an intermediate value, will be the multipliers sought.

120. *Example*. Let the four magnitudes be 3, 2; 11, 8: the first ratio being that of 3 to 2, the latter ratio that of 11 to 8; therefore the first fraction is $\frac{3}{2}$, the latter fraction is $\frac{11}{8}$. Let now these fractions be reduced to two others of the same value, but having the

the ſame denominator, and we have $\frac{3}{2} = \frac{24}{16}$ and $\frac{11}{8} = \frac{22}{16}$. Now between $\frac{24}{16}$ and $\frac{22}{16}$ there lies the fraction $\frac{23}{16}$, being one, out of an infinite number, of an intermediate value. The multipliers therefore may be, *firſt* 3 and 2. *Secondly* 11 and 8. *Thirdly* 23 and 16. We will try all theſe multipliers.

Firſt 3×2 and 11×2 give 6 and 22.

Again 2×3 and 8×3 give 6 and 24.

Therefore the multiple of the firſt term is 6, the multiple of the ſecond 6, that of the third 22, that of the fourth 24. Here the multiples of the firſt and ſecond terms are equal; but the multiple of the third term is leſs than the multiple of the fourth; therefore the latter ratio (that of 11 to 8) is leſs than the former ratio.

Secondly 3×8 and 11×8 give 24 and 88.

Again 2×11 and 8×11 give 22 and 88.

The multiple of the firſt term is 24, of the ſecond term 22, of the third term 88, of the fourth term 88. Here the multiple of the firſt term is greater than the multiple of the ſecond; but the multiple of the third term is equal to (not greater) than the multiple of the fourth; therefore the ratio of 3 to 2, is greater than the ratio of 11 to 8.

Thirdly 3×16 and 11×16 give 48 and 176.

Again 2×23 and 8×23 give 46 and 184.

The multiple of the firſt term is 48, of the ſecond term 46, of the third term 176, of the fourth term 184. Here the multiple of the firſt term is greater than the multiple of the ſecond; but the multiple of the third term is leſs than the multiple of the fourth; therefore the ratio of 3 to 2, is greater than the ratio of 11 to 8.

121. The fractions affording theſe multipliers were

$\frac{3}{2}$, $\frac{23}{16}$ and $\frac{11}{8}$, or rather $\frac{24}{16}$, $\frac{23}{16}$, $\frac{22}{16}$. None out of these limits will afford a proper test; for instance, take $\frac{25}{16}$, that is, take 25 and 16 for the multipliers:

Then, 3×16 and 11×16 give 48 and 176.
Also, 2×25 and 8×25 give 50 and 200.

Here the multiple of the first term, or 48, is less than the multiple of the second, or 50; and the multiple of the third term, or 176, is also less than the multiple of the fourth, or 200; yet for all this the ratio of 3 to 2, is neither less nor equal to the ratio of 11 to 8.

Once more: take for the fraction $\frac{21}{16}$, that is, make the multipliers 21 and 16.

Then, 3×16 and 11×16 give 48 and 176.
Also, 2×21 and 8×21 give 42 and 168.

Here the multiple of the first term, or 48, is greater than the multiple of the second, or 42: and the multiple of the third term, or 176, is also greater than the multiple of the fourth, or 168; yet we are not to conclude, either that the ratios of 3 to 2, and of 11 to 8, are equal, or that the latter is greater than the former, because there ARE multipliers to be found such, that when the multiple of the first is greater than that of the second; the multiple of the third is NOT greater than the multiple of the fourth: and when this is *possible*, then the first term is said to have to the second, a greater ratio than the third term has to the fourth.

122. DEF. 8. ought to be left out, as unnecessary after definition 6.

123. DEF. 9. When proportion is said to consist of three terms, it is meant that the middle term should be repeated, so as in fact to make four, only the two middle terms are the same; the consequent of the first ratio being made the antecedent of the second ratio; see

ſee par. 117. Def. A. (in Simſon) will be conſidered in par. 141.

The doctrine of PROPORTIONALITY, as reſtrained to numbers and commenſurable quantities; and its connexion with the principles laid down in El. 5th.

124. The definition of proportion in numbers given by Euclid (Def. 20. VII. El.) is this:

Four numbers are proportional when the firſt is the ſame multiple, part, or parts of the ſecond, as the third is of the fourth.

To find what multiple, part, or parts (in Euclid's ſenſe), the firſt is of the ſecond, we muſt divide the firſt by the ſecond. Now by what was ſhown in the doctrine of fractions, par. 18, the quotient of one number, divided by another, may be expreſſed by a fraction, whoſe numerator is the dividend, and denominator the diviſor: therefore the criterion of proportionality in numbers may ſtand thus.

Four numbers are proportional, when a fraction whoſe numerator is the firſt, and denominator the ſecond term; is equal to a fraction whoſe numerator is the third, and denominator the fourth term.

125. This criterion may be extended to other quantities beſides numbers, with theſe two proviſoes. Firſt, that the firſt and ſecond terms be of the ſame kind (as in Def. 4. par. 108.); otherwiſe the firſt cannot be any multiple, part, or parts of the ſecond. In ſuch a caſe it would be abſurd to ſay, the ſecond is contained any number of times in the firſt, or to talk of the quotient of the firſt divided by the ſecond; for ſuch quotient only expreſſes the number of times the ſecond is contained in the firſt. Thus it would be abſurd to ſay, a yard in length is contained any number of times whatever in an hour of duration; or to enquire, how often a yard in length, can be found in an hour of time; as is done in diviſion of abſtract

num-

numbers. For the ſame reaſon, the third and fourth terms muſt be homogeneous (to each other) though they may be of a different kind from the firſt and ſecond terms. The ſecond proviſo is, that the firſt term muſt be commenſurable with the ſecond; and the third term commenſurable with the fourth; otherwiſe the criterion abovementioned cannot be applied. When two quantities are commenſurable, they are as number to number, as was obſerved, par. 105. But if they have no common meaſure, they are not as number to number; the one is not exactly any multiple, part, or parts of the other, ſee par. 106; the quotient of one divided by the other, cannot be aſſigned or expreſſed by any number whatever, whether integral or fractional; and therefore the fraction whoſe numerator is the firſt term, and denominator the ſecond, cannot be aſſigned, nor any thing affirmed about its equality to another fraction. For this reaſon Euclid has given a different criterion of proportionality. (Def. 5. V. El.) as was before obſerved, par. 109.

126. It will be proper to ſhow that theſe two definitions are conſiſtent with each other; or that when four (commenſurable) quantities are proportional, according to the 5th definition of Euclid, they are alſo proportional according to the definition in par. 124; and the converſe.

127. I ſay then, if a is to b, as c is to d, according to Euclid's 5th El. that $a \times d = b \times c$.

Firſt,	$a : b :: c : d$	by the hypotheſis.
	$a : b :: a \times d : b \times d$	15. V. El.
	$c : d :: a \times d : b \times d$	11. V. El.
	$c : d :: b \times c : b \times d$	15. V. El.
	$a \times d : b \times d :: b \times c : b \times d$	11 V. El.
	$a \times d = b \times c$	9 V. El.

128. Cor. $\frac{a}{b} = \frac{c}{d}$; that is, if four quantities are proportional, according to Euclid's definition, V. El. they are alſo proportional according to the definition in par. 124.

129. *Converſely.* If four quantities are proportional according to par. 124, they are alſo proportional according to V. El. which may be thus ſhown.

If $\frac{a}{b}=\frac{c}{d}$, then is $a \times d = c \times b$; but if $a \times d = c \times b$, then is a to b, as c to d, according to definition 5th V. El. For let any number, as 3, be the common multiplier of a and c; and let any other number, as 2, be the common multiplier of b and d; I ſay, if $3a$ is greater than $2b$, then $3c$ is greater than $2d$; if equal, equal; if leſs, leſs. For becauſe $a \times d = b \times c$, it follows that $3a \times 2d = 2b \times 3c$; therefore if $3a$ be greater than $2b$, then $3c$ is greater than $2d$; if equal, equal; if leſs, leſs. The ſame is true, whatever be the multipliers: therefore a is to b, as c is to d; according to definition 5th, V. El.

PROPERTIES OF PROPORTIONAL NUMBERS.

130. The criterion of proportionality in numbers being in ſo many caſes applicable to OTHER QUANTITIES, it may be proper here to demonſtrate algebraically, the chief properties of proportional numbers, as theſe properties will hold good in all caſes where that criterion can be applied: ſee par. 125.

131. Now if $a : b :: c : d$; that is, if $\frac{a}{b}=\frac{c}{d}$ (according to the definition, par. 124.) then is $a \times d = b \times c$; that is, if four proportional quantities be expreſſed in numbers, then is the product of the multiplication of the two extreme terms, equal to the product of the multiplication of the two mean terms. Converſely, if $a \times d = b \times c$, then is $\frac{a}{b}=\frac{c}{d}$, or $a : b :: c : d$: that is, if four quantities are of ſuch a ſort, that when expreſſed in numbers, the product of the multiplication of the extremes, is equal to the product of the means, then are thoſe four quantities proportional. This is an algebraic proof of the 16th of VI. El. analogous to

thoſe

those before given of the propositions of the second book: see par. 74.

Cor. 1. Hereafter we may take either equation, viz. $\frac{a}{b}=\frac{c}{d}$, or $a \times d=b \times c$, for the mark of proportionality; one equation implying the other.

Cor. 2. If the two middle terms are alike; that is, if $a : b :: b : c$, then is, $a \times c=b \times b$; that is, the product of the extremes is equal to the square of the means: and conversely, if $a \times c=bb$, then is $\frac{a}{b}=\frac{b}{c}$; that is, $a : b :: b : c$. This answers to 17. VI. El.

Cor. 3. If $a \times d=b \times c$, then $d=\frac{b \times c}{a}$, which is the demonstration of the common rule of three; that is, to find a fourth proportional in numbers, you must multiply the second and third terms together, and divide that product by the first.

Cor. 4. Equal quantities have the same ratio to the same quantity; that is, if a and b be equal, and c be a third quantity, then a is to c, as b is to c: for if $a=b$, then $\frac{a}{c}$ and $\frac{b}{c}$ are equal fractions. This is the 7th of V. El. Also, if a and b have each of them the same ratio to c, that is, if a is to c, as b is to c, then are a and b equal: for if $a : c :: b : c$, then is $\frac{a}{c}=\frac{b}{c}$; multiply both sides of the equation by c, and we have $a=b$. This is the 9th of V. El.

Cor. 5. If two ratios are equal to a third, they are equal to one another; that is, if a be to b, as c is to d, and c is to d, as e is to f, then a is to b, as e is to f: for if $\frac{a}{b}=\frac{c}{d}$, and $\frac{c}{d}=\frac{e}{f}$, then is $\frac{a}{b}=\frac{e}{f}$, therefore $a : b :: e : f$. This is the 11th of V. El.

Cor. 6. Quantities have the same ratio to one another, which their equimultiples have; that is, a is to

to b, as $r \times a$ is to $r \times b$; for $\frac{r \times a}{r \times b} = \frac{a}{b}$ by the doctrine of fractions, therefore $a : b :: r \times a : r \times b$. This is the 15th, V. El.

Alſo a is to b, as $\frac{a}{r}$ is to $\frac{b}{r}$: for the equimultiplier is $\frac{1}{r}$. Otherwiſe we may ſay that $a \times \frac{b}{r} = b \times \frac{a}{r}$, and therefore $a : b :: \frac{a}{r} : \frac{b}{r}$.

Moreover: if $a : b :: c : d$, then $r \times a : r \times b :: s \times c : s \times d$; for $\frac{r \times a}{r \times b} = \frac{a}{b}$, and $\frac{s \times c}{s \times d} = \frac{c}{d}$, but $\frac{a}{b} = \frac{c}{d}$ by the hypotheſis; therefore $\frac{r \times a}{r \times b} = \frac{s \times c}{s \times d}$ and $r \times a : r \times b :: s \times c : s \times d$, by the definition, par. 124.

Further: If $a : b :: c : d$, then, multiplying the antecedents by r, and the conſequents by s, we have $r \times a : s \times b :: r \times c : s \times d$; for by the hypotheſis $\frac{a}{b} = \frac{c}{d}$, therefore $\frac{r}{s} \times \frac{a}{b} = \frac{r}{s} \times \frac{c}{d}$, and $r \times a : s \times b :: r \times c : s \times d$, by par. 124.

Cor. 7. If four proportional quantities are expreſſed in numbers, then are the ſquares of thoſe numbers alſo proportional; and ſo alſo are any other quantities having the proportion of thoſe ſquare numbers; ſee par. 75. That is, if $a : b :: c : d$, then is, $aa : bb :: cc : dd$; for if $a \times d = b \times c$, then $aa \times dd = bb \times cc$, and $aa : bb :: cc : dd$. This anſwers to 22. VI. El.

Cor. 8. If four quantities are proportional, they are proportionals when taken inverſely: that is, if $a : b :: c : d$, then *inverſely* $b : a :: d : c$; for in this caſe alſo $a \times d = b \times c$. This is prop. B. and follows the 6th of V. El. in Simſon.

Cor. 9. If four quantities are proportional, they are alſo proportional when taken alternately. *But here*

 all

all four quantities must be of the ſame kind, *otherwiſe this is not true; though it is of the abſtract numbers, by which thoſe quantities are repreſented.* I ſay then, *if* $a : b :: c : d$; that by *permutation*, or *alternately*, $a : c :: b : d$; for in this caſe alſo $a \times d = b \times c$. This is the 16th of V. El.

COR. 10. If four quantities are proportionals, they are ſo when taken jointly; that is, if $a : b :: c : d$, then by *compoſition* (as it is called) $a+b : b :: c+d : d$; for $\overline{a+b} \times d = \overline{c+d} \times b$, or $ad+bd = bc \times bd$, as is evident, becauſe $ad = bc$ by the hypotheſis. This is the 18th of V. El. In the ſame way alſo $\overline{a+b} : a :: \overline{c+d} : c$; for $\overline{a+b} \times c = \overline{c+d} \times a$, for the ſame reaſon as before.

COR. 11. If four quantities are proportionals, they are ſo when taken disjointly; that is, if $a : b :: c : d$, then by *diviſion* (as it is called) $\overline{a-b} : b :: \overline{c-d} : d$; for $\overline{a-b} \times d = \overline{c-d} \times b$, or $ad - bd = bc - bd$, becauſe, as before, $ad = bc$ by the hypotheſis. This is the 17th of V. El.

COR. 12. If four quantities are proportionals, they are ſo by *converſion*; that is, if $a : b :: c : d$, then $a : \overline{a-b} :: c : \overline{c-d}$; for $\overline{c-d} \times a = \overline{a-b} \times c$, or $ac - ad = ac - bc$, becauſe, as before, $ad = bc$. This is prop. E. and follows the 19th of V. El. in Simſon.

COR. 13. If $a : b :: c : d$, then $\overline{a+b} : \overline{a-b} :: \overline{c+d} : \overline{c-d}$; for $\overline{a+b} \times \overline{c-d} = \overline{a-b} \times \overline{c+d}$, or $ac+bc-ad-bd = ac-bc+ad-bd$; and this will be true if $bc - ad = ad - bc$, or $2bc = 2ad$, but this is true, becauſe $ad = bc$ by the hypotheſis. This, though not in Euclid, is an uſeful propoſition.

COR. 14. If ſeveral homogeneous quantities, when taken two and two, have each a *given* ratio (that is, one and the ſame known ratio) to one another: then, as the antecedent of that given ratio, is to its conſequent, ſo is the ſum of all the other antecedents, to the ſum of all the other conſequents.

Let the given ratio be that of a to b; let the other quan-

quantities be c and d; e and f; g and h; then by the hypothesis, $a : b :: \left\{\begin{array}{l} c : d \\ e : f \\ g : h \end{array}\right\}$ whence $\left\{\begin{array}{l} ad = bc \\ af = be \\ ah = bg \end{array}\right.$

and summing up all these equations we have $ad + af + ah = bc + be + bg$, or $a \times \overline{d + f + h} = b \times \overline{c + e + g}$; therefore $a : b :: c + e + g : d + f + h$. If we sum up the terms a and b with the others (stating the first analogy thus, $a : b :: a : b$, and the first equation $ab = ba$) then this proposition coincides with the 12. V. El. But the proposition is most useful in the form here laid down.

COR. 15. If there be a series of three or more quantities, a, b, c; also another series of as many quantities, d, e, f; and if the first pair in the former, are proportional to the first pair in the latter; the second pair in the former, to the second pair in the latter, and so on *in order*; then are the extreme terms in the former series, proportional to the extreme terms in the latter.

Thus, if $\left.\begin{array}{r} a : b :: d : e \\ \text{and } b : c :: e : f \end{array}\right\}$ whence $\left\{\begin{array}{l} a \times e = b \times d \\ b \times f = c \times e \end{array}\right.$

Then is $a : c :: d : f$; for from the first of the above equations we have $\frac{a}{d} = \frac{b}{e}$; and from the latter $\frac{b}{e} = \frac{c}{f}$; whence $\frac{a}{d} = \frac{c}{f}$ and $a \times f = c \times d$, that is, $a : c :: d : f$. This is called *ex æquo ordinate*, and is the 22. V. El.

COR. 16. If the proportion be not (as before) between correspondent pairs, but between any others; as if the first pair in the former series, be proportional to the second pair in the latter; and the second pair in the former series, be proportional to the first pair in the latter; then also are the extreme terms in each series proportional.

Thus, if $\left.\begin{array}{r} a : b :: e : f \\ \text{and } b : c :: d : e \end{array}\right\}$ whence $\left\{\begin{array}{l} a \times f = b \times e \\ b \times e = c \times d \end{array}\right.$

Then is $a : c :: d : f$; for, from the equations above, it

it is evident that $a\times f=c\times d$: whence $a:c::d:f$. This is called *ex æquo perturbate*, and is the 23. V. El.

SCHOLIUM. The propositions in the fifth book of the Elements mostly referred to, are the 7th, 9th, 11th and 15th. These are demonstrated in cor. 4, 5, 6. The 16th and 17th of VI. El. are very much referred to: these are demonstrated in par. 131. and cor. 1, 2; only all these demonstrations are subject to the restrictions mentioned in par. 125.

Those who will not be at the trouble of studying the whole fifth book, may read the definitions only; with what is here delivered about them and about proportionals, and then pass on to the sixth book; the first and last of which are here demonstrated in an easier way.

After the first 17 propositions of the sixth book, the reader may take the 19th, 30th and 33d, omitting the rest as of less general use: or if he thinks this too much, he may read only the eight first propositions and the last; but these are absolutely necessary for every one who means to study either astronomy or natural philosophy.

BOOK THE SIXTH.

132. PROP. I. We will here give another demonstration of the former part of this proposition; restrained indeed to commensurables, but perhaps on that very account easier to be understood by learners.

Let the triangles *ACH* and *ADL* have the same altitude, viz. the perpendicular drawn from the point *A* to *BD*; then as the base *CH* is to the base *DL*, so is the triangle *ACH* to the triangle *ADL*.

Let the base *CH* be commensurable to the base *DL*, and let *CB* be their common measure. Divide the base *CH* by the common measure into the equal parts *CB*, *BG*, *GH*; also divide the base *DL* by the same common measure into the parts *DK*, *KL*, equal to one another, and to each of the parts of the base *CH*.

From

From the vertex *A*, draw lines to each of the points of diviſion in the baſe *CH*; that is, draw *AB* and *AG*, and the great triangle *ACH* will be divided into as many leſs triangles, as there are parts in its baſe *CH*. In like manner, draw lines to each of the points of diviſion in the baſe *DL* (viz. draw *AK*) and this triangle alſo will be divided into as many leſs triangles, as there are parts in its baſe *DL*. Now all theſe leſs triangles, in each of the two greater ones, are equal to one another by 38. I. El. Therefore the whole triangle *ACH* will be to the whole triangle *ADL*, as the number of little triangles which make up the former, to the number of little triangles which make up the latter; that is, as the number of equal parts (or lines) which make up the baſe of the former triangle, to the number of (the like) equal parts which make up the baſe of the latter; that is, as the whole baſe *CH* to the whole baſe *DL*.

The latter part of the demonſtration, " And becauſe the parallelogram," &c. may ſtand as in Euclid.

133. Prop. 4. " The ſides about equal angles are proportionals," and " the ſides oppoſite to equal angles are homologous."

Thus the angle at *A* being equal to the angle at *D*, the ſides about the former are proportional to the ſides about the latter; that is, *AB* : *AC* :: *DC* : *DE*. And the two antecedents *AB* and *DC*, are oppoſite to the equal angle at *C* and *E*; alſo the two conſequents *AC* and *DE* are oppoſite to the equal angles at *B* and *C*. See Def. 12. V. El.

134. In equiangular triangles, the ſides oppoſite to equal angles, are by ſome called *correſponding ſides*; and correſponding ſides in equiangular triangles are ſaid to be proportional: that is, as the ſide *AB* in one ſuch triangle, is to its correſponding ſide *DC* in the other; ſo is the ſide *AC* in the former triangle, to its correſponding ſide *DE* in the latter (ſee Mr. T. Simpſon's Geometry, B. IV. p. 14.). That this is

true is evident; for it is only taking Euclid's proportionals alternately, by 16. V. El.

Though the term *homologous* has reference to a particular order in Euclid's way of ſtating theſe proportionals; yet homologous ſides are alſo correſponding ſides, both being thoſe that are oppoſite to equal angles.

135. It follows from this propoſition, that equiangular triangles are of neceſſity *ſimilar figures*, according to Def. 1. VI. El.: therefore in ſpeaking of triangles, *ſimilar triangles* is often put to denote *equiangular* triangles.

136. Prop. 18. Similar figures have been defined, but not ſimilarity of ſituation. Similar figures are alſo ſimilarly ſituated, when all their homologous ſides are parallel. Thus, becauſe *AB* is to *AG*, as *CD* to *CF*, therefore the homologous ſides *AB* and *CD* muſt be parallel, and alſo the homologous ſides *AG* and *CF* muſt be parallel. Moreover, becauſe *GH* : *HB* :: *FE* : *ED*, therefore alſo *GH* muſt be parallel to *FE*, and *HB* parallel to *ED*.

Although ſimilarity of ſituation is mentioned as a condition in the enunciation of the problem, yet it ſeems not noticed in the conſtruction: no notice is taken that the line *AB*, given in length, muſt be of neceſſity parallel to one of the ſides of the given rectilinear figure; and in the concluſion it is affirmed, that the figure made in conſequence of the conſtruction laid down, is indeed deſcribed upon the given ſtraight line, and is ſimilar to the given figure; but no notice is taken of its ſituation with reſpect to that given figure.

137. Prop. 27. The 27th of the ſixth book is introduced for the ſake of the 28th; and the two problems in the 28th and 29th (as Dr. Simſon ſays) for the ſake of two other problems, viz. Having given, either the ſum or the difference of the two ſides of a right angled parallelogram, and its magnitude (that is, the ſide of a ſquare equal to it), to find the

the ſides of that parallelogram. Or, which is the ſame thing, Having given the ſum or difference of two ſtraight lines, and the mean proportional between them; to find the lines themſelves. Both theſe problems have been ſolved algebraically, and conſtructed geometrically in the former part (par. 272 and 279): nor can we think theſe intricate propoſitions, relative to parallelograms, deficient and redundant, ſo uſeful as Dr. Simſon repreſents them.

138. PROP. 30. In the 11th prop. of the ſecond book, we are taught "to divide a given ſtraight line into two parts, ſo that the rectangle contained by the whole and one of the parts (viz. the leſs part) ſhall be equal to the ſquare of the other part (viz. the greater part *)." In this propoſition we are taught to divide the ſame given ſtraight line into two parts, ſo that the whole line ſhall be to the greater part, as the ſame greater part is to the leſs, or, as it is called in Def. 3. VI. El., to divide a given ſtraight line in *extreme and mean ratio*. Now, from the 17th of the VI. El. we learn, that when three ſtraight lines are thus proportionals, the rectangular figure contained by the extremes, is equal to the ſquare figure deſcribed upon the mean; therefore this propoſition is in effect the ſame with the 11th of II. El. In that propoſition (11. II. El.) the whole relates to the agreement of one figure with another; taking figure in the ſenſe of definition 14. I. El. The problem is, how to make a certain rectangular figure there deſcribed, equal to a ſquare, and the idea of ratio is never introduced. In this propoſition (30. VI. El.) the whole relates to the agreement of certain ratios; and the problem is, how to divide the line ſo, that the ratio which the whole line bears to the greater part, may be equal to the ratio which the greater part bears to the leſs; and the idea of figure (Def. 14. I. El.) never enters into the problem.

* For the rectangle under the whole, and the greater part, exceeds the ſquare of the greater part; much more does it exceed the ſquare of the leſs part.

problem. It is true, this idea is introduced into the ſolution; but it is immediately dropped, when by that means we have obtained the ratio which the ſides of thoſe figures (ſo introduced) bear to one another. And we cannot help thinking the ſolution would have been more natural, had it proceeded like the 13th of VI. El. and the idea of *figure* had never entered into it at all. The following ſolution from Mr. T. Simpſon is of this ſort.

139. Let AB (fig. 11.) be the given line, make BC perpendicular to it (11. I. El.) and equal to $\frac{1}{2}AB$; on the center C and with the radius CB, deſcribe a circle. Draw $AGCH$ through the center C, cutting the circle in G and H. Take $AE = AG$, and the line AB is divided in E in extreme and mean ratio.

For AB being a tangent to the circle in B (16. III. El.) we have $AH \times AG = AB^2$ (36. III. El); whence $AH : AB :: AB : AG$ (17. VI. El.), whence by inverſion $AB : AH :: AG : AB$; and by converſion $AB : AH - AB :: AG : AB - AG$. Now $AB = 2BC = GH$ and $AH - AB = AH - GH = AG = AE$; alſo $AB - AG = AB - AE = BE$; therefore $AB : AE :: AE : BE$; therefore AB is divided in E in extreme and mean ratio.

Scholium. This conſtruction entirely agrees with that given by Euclid, prop. 11. II. El. AB here anſwers to AB there; BC to AE, and AC to BE or EF; of courſe AG anſwers to AF and AE to AH; and the ſame method of calculation may be inferred from this as from that conſtruction.

140. Prop. 33. We ſhall demonſtrate the former part of this propoſition, in the ſame manner as the former part of prop. 1. VI. El. was demonſtrated in par. 132, and for the reaſon there given.

Let $BCKL$ and FMN be equal circles; let the angles at their centers be BGL and FHN: then as the circumference BL is to the circumference FN, ſo is the angle BGL to the angle FHN.

These

These circumferences being supposed commensurable, let *BC* be their common measure; divide the circumference *BL* by their common measure into the equal circumferences or parts, *BC*, *CK*, *KL*. Also, divide the circumference *FN*, by the same common measure, into the parts *FM*, *MN*, equal to one another, and also to each of the parts of the circumference *BL*. From the center G, draw lines to each of the points of division in the circumference *BL*; that is, draw *GC* and *GK*, and the whole angle *BGL* will be divided into as many less angles, as there are parts in the circumference *BL*. In like manner, from the center *H*, draw lines (*HM*) to each of the points of division in the circumference *FN*, and the angle *FHN* also will be divided into as many less angles, as there are parts in the circumference *FN*. Now all these less angles in each of the two greater ones are equal to one another, by 27. III. El.; therefore the whole angle *BGL* will be to the whole angle *FHN*, as the number of little angles that make up the former, to the number of little angles that make up the latter; that is, as the number of equal parts, or equal circumferences, which make up the former circumference, to the number of like parts which make up the latter; or as the whole of the former circumference *BCKL* is to the whole of the latter circumference *FMN*; therefore, as the circumference *BL* is to the circumference *FN*, so is the angle *BGL* to the angle *FHN*.

Of the Composition and Resolution of Ratios.

141. Definition. If there be a series of any quantities of the same kind, as *A*, *B*, *C*, *D*, *E*, &c. then the ratio of the extremes is said to be compounded of all the intermediate ratios; that is, the ratio of *A* to *E*, is said to be compounded of the ratios of *A* to *B*, of *B* to *C*, of *C* to *D*, and of *D* to *E*, the consequent of one ratio being the antecedent of the next. See Def. A. V. El. in Simson.

For

For an illustration of this definition see the latter part of Simson's note on prop. 23. VI. El. beginning at page 322.—*But no body, &c.*

142. THEOREM. The ratio compounded of the several ratios of A to B, of C to D, and of E to F, expressed in numbers, is that of $A \times C \times E$ to $B \times D \times F$, or the product of the multiplication of all the antecedents, to the product of the multiplication of all the consequents.

For $A : B :: ACE : BCE$ (15. V. El.) and $C : D :: BCE : BDE$; also $E : F :: BDE : BDF$. But in the series of quantities $ACE : BCE : BDE : BDF$, the ratio of the extremes, ACE to BDF is compounded of all the several intermediate ratios by par. 141, that is to say (by 11. V. El.), of the several ratios of A to B, and C to D, and E to F.

143. PROBLEM I. To resolve the ratio of E to F (expressed in numbers) into two other ratios, whereof one is the given ratio of A to B, the other such as being compounded with the given ratio of A to B, shall make the proposed ratio of E to F.

Let the ratio sought be that of R to S; then by the conditions of the problem $AR : BS :: E : F$, therefore by par. 131, $ARF = BSE$; whence $R : S :: BE : AF$; that is, the ratio sought is that of $B \times E$ to $A \times F$. Hence

RULE. Multiply the antecedent of the former given ratio (viz. that which is to be resolved into two others) by the consequent of the latter given ratio, for the antecedent of the ratio sought; and multiply the consequent of the former ratio, by the antecedent of the latter, for the consequent of the ratio sought.

Example. Resolve the ratio of 3 to 2 into two others, and let one of them be the ratio of 4 to 3; What is the other? Answer, That of 3×3 to 2×4, or the ratio of 9 to 8.

Proof. The ratio of 4 to 3, compounded with that of 9 to 8, makes the ratio of 4×9 to 3×8, or

that of 36 to 24, or that of 12×3 to 12×2, or that of 3 to 2.

144. Prob. 2. Two ratios (expreſſed in numbers) being propoſed, to find two other ratios reſpectively equal to the former two, but having a common conſequent.

Rule. Multiply the terms of each ratio by the conſequent of the other, and you will have two ratios equal to the propoſed ratios, and having a common conſequent.

Let the two propoſed ratios be that of 5 to 4, and 7 to 3: then the former ratio is equivalent to that of 5×3 to 4×3, and the latter is equivalent to 7×4 to 3×4; that is, the two ratios ſought are that of 15 to 12, and 28 to 12.

That the latter ratios are reſpectively equal to the former, appears from par. 131. cor. 6.: that they will have the ſame conſequent is evident; becauſe the product of the multiplication of two numbers is the ſame, which ever of the two is made the multiplier: ſo here $4 \times 3 = 3 \times 4$.

145. When ratios have thus one common conſequent, we may then compare them as to magnitude; and here the latter ratio, viz. that of 28 to 12, is greater than the former, viz. that of 15 to 12, by prop. 8. V. El.

146. When a greater quantity is compared with a leſs, the ratio reſulting therefrom is called a ratio *majoris inæqualitatis*. When a leſs quantity is compared with a greater, this is called a ratio *minoris inæqualitatis*; and the common boundary between them is the ratio *æqualitatis*.

147. If a ratio majoris inæqualitatis be compounded with another ratio it increaſes that ratio; thus, if the ratio of 2 to 1 be compounded with that of 6 to 1, it makes the ratio of 12 to 1; and this is a greater ratio that that of 6 to 1. But if a ratio minoris inæqualitatis be compounded with another ratio it diminiſhes that ratio; thus, if the ratio of 1 to 2 be com-

compounded with that of 6 to 1, it makes the ratio of 6 to 2, or of 3 to 1; and this is a leſs ratio than that of 6 to 1. If the ratio æqualitatis be compounded with another ratio it neither increaſes nor diminiſhes that ratio; thus, if the ratio of 3 to 3 be compounded with that of 6 to 1, it makes the ratio of 18 to 3, or of 6 to 1, as before.

148. There is therefore an analogy between the effects of theſe ratios in compoſition and the effects of poſitive and negative quantities, in what we called the algebraical incorporation of quantities. Hence, ratios majoris inæqualitatis have been ſtyled *affirmative ratios*, and the ratios minoris inæqualitatis, *negative ratios*. And as the incorporation of quantities in algebra is called their *addition*, ſo the compoſition of ratios is ſometimes called the *addition* of ratios.

149. The analogy is carried on ſtill further; the buſineſs of algebraic ſubtraction is to find ſuch a quantity, as being algebraically added to the ſecond of two propoſed quantities, will produce the firſt; and the buſineſs of the problem in par. 143. is to find ſuch a ratio, as being compounded with the latter of two propoſed ratios, or that of A to B, ſhall produce the former, or that of E to F. And therefore thoſe who call the compoſition of ratios, the addition of ratios; will ſay, that the problem there propoſed is this; "To *ſubtract* the given ratio of A to B from "the given ratio of E to F;" and that, as in algebra we are to change the ſign of the ſubtrahend, and then add it to the firſt quantity; ſo here we are to invert the terms of the ratio to be ſubtracted, and then compound it with the former ratio. Thus, to ſubtract the ratio of A to B, from the ratio of E to F; invert the terms of the ſubtrahend, and it becomes the ratio of B to A; compound (or add) this to the ratio of E to F, and its makes the ratio of $B \times E$ to $A \times F$, conſonant to what was ſhown in par. 143. And in this way it may be ſaid, that the *difference* of the

the two proposed ratios, is the ratio of $B \times E$ to $A \times F$*.

150. Multiplication is only a repeated addition of the same quantity. Therefore, if the composition of two ratios, be called their addition; the repeated composition of the same ratio must be called, the *multiplication* of that ratio; the multiplier signifying how many times it is so compounded. Thus, the ratio of A to B (expressed in numbers) if compounded with itself (that is, with the ratio of A to B) makes the ratio of $A \times A$ to $B \times B$, or A^2 to B^2; therefore the doubled, or duplicate ratio of A to B, is that which the square of A has to the square of B. In like manner, the triplicate ratio of A to B, is that which A^3 has to B^3. And in general, if the ratio of A to B, be compounded with itself n times, it makes the ratio of A^n to B^n. In other words, if the ratio of A to B be taken n times, or to be *multiplied* by n, it makes the ratio of A^n to B^n.

151. And to keep up the analogy, the nth part of the ratio of A^n to B^n, is the ratio of A to B; or, we may say, the ratio of A^n to B^n, *divided* by the number n, gives the ratio of A to B. And so in the same way, the half of the ratio of A to B, is that of $\sqrt{A}$ to $\sqrt{B}$; or, which is a more common form of speaking, the subduplicate ratio of A to B, is that which the square root of A has to the square root of B.

Of the Newtonian definition of the word AS.

152. No expression occurs so frequently in books of philosophy as this, "The quantity A is directly *as*

* There are who say, there is much more in this than mere analogy. That ratios are quantities *sui generis*; that is, have a nature of their own; and are by that nature, as capable of addition, subtraction, &c. as either numbers or lines. Dr. Simson in his note on 23. VI. El. has shown, that Euclid does not so consider the matter.

the

the quantity B; or A is *as* B." Newton has a ſcholium purpoſely to explain this uſe of the word *as*, which we ſhall now enlarge upon, and illuſtrate by examples.

153. DEFINITION I. Let there be two quantities (either of the ſame or different kinds) but each of them variable; then if theſe two quantities are ſo connected that one of them cannot be increaſed or diminiſhed, but the other muſt alſo be increaſed or diminiſhed in the ſame ratio, then one of theſe quantities is ſaid to be *directly as* the other; or ſimply *as* the other. Thus, if a ſum of money is to be divided equally among a certain number of men, then each man's ſhare is *as* the whole ſum to be divided. For that whole ſum cannot be doubled, tripled, halved, &c. but each man's ſhare will of neceſſity be alſo doubled, tripled, halved, &c. that is, changed in the ſame proportion. And in general, let A and B be two variable quantities; let A be changed into a, and in conſequence thereof, let B be changed into b. If A is to a, in all caſes, as B is to b, then is A ſaid to be *as B directly*.

Here it muſt be obſerved, that though two quantities only, A and B, are ſpoken of, yet four proportionals are always implied, to wit, that $A : a :: B : b$.

154. If A is as B, and B is as C (in this ſenſe), then A is as C. Let the little letters repreſent the cotemporary changes of theſe variable quantities; then $A : a :: B : b$ by the hypotheſis, and $B : b :: C : c$ by the hypotheſis alſo; therefore $A : a :: C : c$, by 11. V. El. but if A is to a, in all caſes, as C to c, then A is *as* C by the definition.

155. In general all the properties of proportional numbers in par. 131, may be applied here alſo. Thus, if A is as B, and n be any given number; then A is as nB: for if A is as B, then $A : a :: B : b$, but by par. 131. cor. 6. (or 15. V. El.) $B : b :: nB : nb$; therefore $A : a :: nB : nb$. So if A is as B, then is A

as

as $20B$. Thus, in the instance before given, if the whole number of pounds to be divided, be doubled, tripled, &c.; then not only the number of pounds in each man's share, will be double, triple, what they (the number of pounds) were before; but also the number of shillings in each man's share will be double, triple, &c. what they (the number of shillings) were before. And thus the number of pounds to be divided, will be *as* the number of shillings in each man's share; or A as $20B$.

For a like reason, B is *as* $20B$; that is, however the number of pounds in any man's share may vary, the number of shillings in his share will vary also; and in the very same proportion.

156. If a quantity A is so connected with two others B and C, that when B alone is changed to b, the quantity A must be changed in the same proportion; and when C alone is changed to c, A must be changed in the same proportion; and if when both B and C are thus changed, A is changed to a; then will A be to a in a ratio compounded of the two ratios of B to b and of C to c, or, by par. 142, A will be to a in the ratio of $B \times C$ to $b \times c$. For since $B : b :: A : \frac{A \times b}{B}$; it follows, that by the change of B to b, A becomes $\frac{A \times b}{B}$: but this quantity is further changed by the change of C, and that in the ratio of C to c, therefore $C : c :: \frac{A \times b}{B} : \frac{A \times b \times c}{B \times C}$, which is $=a$, the final value of A. From that equation we have $A \times b \times c = a \times B \times C$, and $A : a :: B \times C : b \times c$, that is, A is to a, in a ratio compounded of the ratios of B to b and of C to c, by par. 142.

In such a case A is said to be *as* B and C, jointly; or A is said to be as B directly and C directly; or as $B \times C$. All which is often expressed thus: "when C is given, then A is as B, and when B is given, A is

as

as C; therefore when neither are given, A is as B and as C jointly *."

157. In the very ſame way, if A is as B directly, as C directly, and alſo as E directly; then A is as $B \times C \times E$.

Example. 600£. is placed out to intereſt at 4 per cent. and ſix years intereſt is due. Alſo 400£. is placed out to intereſt at 5 per cent. and ſeven years intereſt is due. What proportion does the intereſt due in one caſe, bear to the intereſt due in the other? It muſt be premiſed, that if we ſuppoſe the rate of intereſt and the time to remain the ſame, and the principal only to vary; then the amount of intereſt due, will be directly as the principal. In like manner, this amount will be (*cæteris paribus*, as it is called) directly as the rate of intereſt. It will alſo be, *cæteris paribus*, directly as the time the money has lain at intereſt; therefore, by what has been ſaid, the proportion ſought is that of $600 \times 4 \times 6$ to $400 \times 5 \times 7$, or $6 \times 4 \times 6$ to $4 \times 5 \times 7$, or 6×6 to 5×7, or as 36 to 35.

All this will appear by computing the real intereſt due; for that in the former caſe is 144£. in the latter 140£.; but $144 : 140 :: 36 : 35$.

158. A. DEFINITION 2. The RECIPROCAL of any number or quantity (arithmetically expreſſed) is unity divided by that quantity; or it is a fraction whoſe numerator is one, and denominator that quantity. Thus, the reciprocal of 10 is $\frac{1}{10}$, the reciprocal of D is $\frac{1}{D}$. Thus alſo $\frac{D}{1}$ and $\frac{1}{D}$ are reciprocal fractions.

158. B. DEFINITION 3. One ratio is ſaid to be the *inverſe* or *reciprocal* of another, when the antecedent of the former is to its conſequent, as the conſequent of the latter is to its antecedent. Thus, the ratio of

* By *given* quantities is generally meant *known* quantities. But in ſpeaking of variable quantities, by *given* is often meant *invariable* or *conſtant* quantities.

3 to 4, is the reciprocal of 4 to 3, or the reciprocal of 8 to 6. And in general, the inverſe ratio of D to d, is the direct ratio of d to D, or the direct ratio of $\frac{1}{D}$ to $\frac{1}{d}$, for $d : D :: \frac{1}{D} : \frac{1}{d}$) or the direct ratio of the reciprocals of D and d.

159. Definition 4. Let there be two quantities, each of them variable, but ſo connected, that one of them cannot be increaſed, but the other muſt be diminiſhed, and that in the ſame ratio in which the former was increaſed: and contrariwiſe; in whatever ratio the former is diminiſhed, muſt the latter be increaſed; then theſe two quantities are ſaid to be *inverſely* or *reciprocally as* one another. Thus, if a certain ſum of money is to be divided equally among a number of men; if we ſuppoſe the number of men to vary, then each man's ſhare will be inverſely as the number of men. The number of men cannot be doubled, tripled, halved, &c. but each man's ſhare will of neceſſity be halved, be one third, or double of what it was before; that is, changed in the contrary or inverſe proportion to that of the number of men.

And in general, let A and D be two variable quantities; let A be changed into a, and in conſequence thereof, let D be changed into d: if A is to a, in all caſes, as d to D, or as $\frac{1}{D}$ to $\frac{1}{d}$, then is A ſaid to be *as D inverſely*.

160. If A is directly as B, and B is inverſely as D; then A is inverſely as D: for $A : a :: B : b$ and $B : b :: d : D$, therefore $A : a :: d : D$; that is inverſely as D to d.

In like manner, if A be reciprocally as D, and D reciprocally as E, then A is directly as E. In a like way may the other properties of proportional numbers be applied here. The little letters repreſenting, as before, the cotemporary values of the quantities repreſented by the great letters.

O 161. If

161. If a quantity A is ſo connected with two others B and D, that when B alone is changed to b, then A muſt be changed in the ſame direct proportion; and when D alone is changed to d, then A muſt be changed in the ſame proportion, but inverſely: and if when both B and D are changed, A becomes a; then will A be to a in a ratio compounded of the direct ratio of B to b, and the inverſe ratio of D to d. For ſince $B : b :: A : \frac{A \times b}{B}$, A becomes $\frac{A \times b}{B}$ in conſequence of the change of B alone. But this quantity is further changed by the change of D, and that in the inverſe ratio of D to d, or in the direct ratio of d to D; therefore $d : D :: \frac{A \times b}{B} : \frac{A \times D \times b}{B \times d}$, which is $= a$. From that equation we have $A \times D \times b = B \times a \times d$, and $A : a :: B \times d : d \times D$; that is, A is to a in a ratio compounded of the direct ratio of B to b, and of the inverſe ratio of D to d, by par. 142.

In ſuch a caſe, A is ſaid to be *as* B directly, and *as* D inverſely. Or it is ſaid that when D is given, A is directly as B, and when B is given, A is inverſely as D; therefore when neither are given, A is directly as B, and inverſely as D.

162. In like manner, if A is directly as B, directly as C, and directly as E, alſo inverſely as D and inverſely as F; then is A to a, as $B \times C \times E \times d \times f$ to $b \times c \times e \times D \times F$, or as $\frac{B \times C \times E}{D \times F}$ to $\frac{b \times c \times e}{d \times f}$. And in ſuch a caſe, A is ſaid to be *as* the fraction $\frac{B \times C \times E}{D \times F}$; that is, ſuppoſing all theſe quantities variable, as before, the value of A at one time, is to the value of A at another, (or A to a) as the value of that fraction in the former caſe, to the value of that fraction in the latter.

163. We may hence alſo find the proportion of any other of theſe varying quantities; for inſtance that

that of D to d: for since $A : a :: \frac{B \times C \times E}{D \times F} : \frac{b \times c \times e}{d \times f}$. Multiply antecedents by D and consequents by d (see par. 131. cor. 6.) and $A \times D : a \times d :: \frac{B \times C \times E}{F} : \frac{b \times c \times e}{f}$, whence $D : d :: \frac{B \times C \times E}{A \times F} : \frac{b \times c \times e}{a \times f}$. All which is commonly represented in a shorter way thus, A is as $\frac{B \times C \times E}{D \times F}$, therefore $A \times D$ is as $\frac{B \times C \times E}{F}$, and D is as $\frac{B \times C \times E}{A \times F}$.

164. *Example.* Let P be the principal sum put out to interest, R the rate of interest, T the time it continues at interest, A the amount of the interest in that time. Then it is evident, from what has been said, that A is as $P \times R \times T$; whence, arguing as before, R is as $\frac{A}{P \times T}$. Suppose then the sum of 600£. continues out at interest 6 years, and the interest in that time amounts to 144£. Suppose also that the sum of 400£. continues out at interest for 7 years, and the interest in that time amounts to 140£. What proportion does the rate of interest in the former case bear to the rate of interest in the latter?

Answer, that of $\frac{144}{600 \times 6}$ to $\frac{140}{400 \times 7}$, or that of $144 \times 400 \times 7$ to $140 \times 600 \times 6$, or that of $144 \times 4 \times 7$ to $140 \times 6 \times 6$, or that of $4 \times 4 \times 7$ to 140, or that of 4×28 to 140, or 4 to $\frac{140}{28}$, or that of 4 to 5. So that if the former sum was out at 4 per cent. the latter was out at 5 per cent.

165. Because R is as $\frac{A}{P \times T}$, if the time be the same in the two cases (or given) then R will be as $\frac{A}{P}$, or di-

rectly as the intereſt due, and inverſely as the principal: for when the time is the ſame, $T=t$. Thus, let the annual intereſt of 600£. be 24£. In another caſe, let the annual intereſt of 400£. be 20£.; What is the proportion of their rate of intereſt? Anſwer, that of $\frac{24}{600}$ to $\frac{20}{400}$ or $\frac{24}{6}$ to $\frac{20}{4}$, or 4 to 5.

166. Further: When T is given, R is as $\frac{A}{P}$. Let now A be as P; that is, let A have always the ſame ratio to P, ſo that $A : P :: a : p$, then $\frac{A}{P}=\frac{a}{p}$; that is, although A changes to a, and P to p, yet $\frac{A}{P}$ is always the ſame, or a given quantity: In this caſe R which is as $\frac{A}{P}$ will be an invariable quantity. We need not an example to illuſtrate this. It is evident, that if the annual intereſt has always the ſame proportion to the principal; the rate of intereſt is always the ſame, whatever that rate may be.

167. Once more; Let P be given, then A is as $R \times T$; let A be given alſo, then R is inverſely as T: for firſt, $A : a :: R \times T : r \times t$. But, ſecondly, A is ſuppoſed invariable, therefore $A=a$, and $R \times T=r \times t$; that is, $R : r :: t : T$, or R is inverſely as T: that is, if both principal and intereſt be the ſame, but the rate and time differ; then the leſs the rate, the longer is the time the money muſt lie to raiſe the ſame intereſt; and contrariwiſe.

168. Theſe caſes may be varied without end: what has been ſaid is ſufficient to explain Newton's ſcholium to Lemma X. Book I. of his *Principia*; which is all that we propoſed.

APPENDIX.

APPENDIX.

Consisting of propositions here subjoined, either for their usefulness, or the peculiar manner of their demonstration.

Tab 11 Geom

169. PROP. I. THEOREM. If a right line *TA* (fig. 12, 13, 14.) touch a circle in the point *A*, and from the point of contact there be drawn the two chords *AB* and *AC*, and from the extremity of one of them, *AC*, there be drawn a line *CD*, parallel to the tangent *TA*, and meeting the other chord (produced if need be) in *D*. Then will the chord *AC* be a mean proportional between the other chord *AB*, and the adjacent segment *AD*; that is, $AC^2 = AB \times AD$.

Join *BC*, and in the triangles *ACB* and *ADC*, the angle $ACB = TAD$ (32. El. III.) $= ADC$ (29. El. I.), and the angle *BAC* is common; therefore the triangles *ACB* and *ADC* are equiangular, and $AB : AC :: AC : AD$ (4. El. VI.), therefore $AC^2 = AB \times AD$ (17. El. VI.)

170. COR. If *AB* (fig, 14.) be a diameter, then will the triangle *ACB* be right angled at *C* (31. El. III.) and the line *CD* will be a perpendicular let down from the right angle to the hypothenuse (18. El. III.). Therefore in any right angled triangle, if a perpendicular be let down from the right angle to the hypothenuse, either leg will be a mean proportional between the whole hypothenuse and the adjacent segment of the hypothenuse, which is prop, 8. VI. El.

171. PROP. 2. THEOREM. If through any point *P* (fig. 16, 17.) either within a circle (case 1st) or without a circle (case 2d) two lines be drawn, so as to cut the circle in the points *A*, *B*, *C*, *D*. I say then, that the rectangle under the parts of each line

cut off by the circle are equal; that is, $PA \times PB = PC \times PD$.

Join the points AD and CB where the two lines cut the circle, and we ſhall have two triangles APD and CPB, whoſe ſides are the parts cut off by the circle. Now the angles BAD and BCD, ſtanding on the ſame arch BD are equal (27. El. III.); and the angles ABC and ADC, ſtanding on the ſame arch AC, are alſo equal (27. El. III.): therefore the triangles APD and CPB are equiangular, and $PA : PD :: PC : PB$ (4. El. VI.) and $PA \times PB = PC \times PD$ (16. El. VI.).

172. Cor. 1. In caſe 1ſt, let AB (fig. 15.) be a diameter, and CD perpendicular to it, then will CD be biſected by the diameter in P (3. El. III.), and $PC = PD$, therefore $PA \times PB = PC^2$ (17. El. VI.); that is, the perpendicular let fall from any point of a circle on the diameter, is a mean proportional between the ſegments of the diameter. Moreover, join AC and BC, and the triangle ACB will be right angled at C (31. El. III.); and AB the hypothenuſe, PC a perpendicular let down from the right angle on the hypothenuſe, which is therefore a mean proportional between the ſegments of the hypothenuſe.

173. Cor. 2. In caſe 2d (fig. 16.) where the point P is without the circle, let one of the lines, as PDC, be conceived to turn round the point P, ſo as to come into the poſition of the line PS, which paſſes through P and touches the circle in S. During this motion the two points of interſection C and D will approach each other, and finally coincide in S. The two parts PC and PD cut off by the circle, will approximate to an equality, and will finally become equal to one another, and to the tangent PS; and the rectangle $PC \times PD$ will finally become equal to PS^2: that is, the rectangle under the two ſegments PA and PB is equal to the ſquare of the tangent PS. See 37. El. III.

174. Scholium. This theorem includes in it the 35th, 36th and 37th of the 3d book, and the 8th of

the 6th book, with its corollary; and the demonstration is far shorter and easier than any of these, but it could not be given before; because it depends on the 4th of the 6th book. Nor could this proposition have been understood, much less demonstrated, till the nature of proportion had been explained, and its properties demonstrated; which is the subject of the 5th book. We shall give another instance how far propositions may be generalized (that is, made to comprehend a number of those we find in Euclid) and yet their demonstration rendered clear and easy, by the help of this most useful truth, that equiangular triangles have their homologous, or their corresponding sides proportional. See 4. VI. El.

175. PROP. 3. THEOREM. In any triangle ACB (fig. 19, 20.) let a perpendicular from the vertex C fall on the base (produced if need be) in D, dividing the base (produced if need be) into two segments DA and DB. I say, that the rectangle under the sum and difference of the sides of the triangle is equal to the rectangle under the sum and difference of the segments of the base; that is, $\overline{AC+CB} \times \overline{AC-CB} = \overline{AD+DB} \times \overline{AD-DB}$.

With the center C and radius $CB =$ the shorter side describe a circle, cutting the longer side, produced in G and H; also cutting the base (produced if need be) in E and B: then (because CG and CH are equal to CB) AH $(=AC+CH=AC+CB)$, is the sum, and AG $(=AC-CG=AC-CB)$ is the difference of the sides AC and CB. Again, because DB and DE are equal (3. III. El.) then in case 1st (fig. 19.) AB is the sum of AD and DB, and AE $(=AD-DE=AD-DB)$ is the difference of AD and DB, the segments of the base. In case 2d (fig. 20.) where the perpendicular falls on the base produced, AE $(=AD+DE=AD+DB)$ is the sum of the segments of the base (produced) and AB $(=AD-DB)$ is the difference of the segments of the base (produced); but $AH \times AG = AB \times AE$ by the last

last par. therefore the rectangle under the sum and difference of the sides of any triangle is equal to the rectangle under the sum and difference of the segments of the base, or of the base produced.

176. COR. 1. Hence if the two sides and base (that is, all three sides) of any triangle be given, the segments of the base may be found: for the base itself is the sum of those segments in the first case, and their difference in the second. Therefore by this proposition it will be, As the base is to the sum of the sides, so is the difference of the sides to the $\left\{\begin{array}{ll}\text{difference,} & \textit{in case } \text{1st} \\ \text{sum,} & \textit{in case } \text{2d}\end{array}\right\}$ of the segments of the base. And in either case, the semi-sum of the base and the fourth proportional so found, will be the greater segment; and their semi-difference, will be the less segment: see par. 215. in the introduction to Algebra.

177. COR. 2. Because $DB = DE$ and $BE = 2BD$, therefore $AB \times AE\ (= AB \times \overline{AB \mp BE})\ = AB \times \overline{AB \mp 2BD} = AB^2 \mp 2AB \times BD$. Hence $\overline{AC + CB} \times \overline{AC - CB} = AB^2 \mp 2AB \times BD$; that is, the rectangle under the sum and difference of the sides of the triangle, is equal to the square of the base $\left\{\begin{array}{l}\text{less } \textit{in case } \text{1st} \\ \text{more } \textit{in case } \text{2d}\end{array}\right\}$ by a double rectangle under the whole base and the less segment BD.

178. COR. 3. When the angle ABC is a right angle (fig. 21.) the points B and D coincide, the line BD vanishes, and the circle touches the base in B; therefore, in a right angled triangle, the rectangle under the sum and difference of the hypothenuse and one of the legs, is equal to the square of the other leg.

COR. 4. By the 2d cor. $\overline{AC + CB} \times \overline{AC - CB} = AB^2 \mp 2AB \times BD$; but $\overline{AC + CB} \times \overline{AC - CB} = AC^2 - CB^2$, therefore $AC^2 - BC^2 = AB^2 \mp 2AB \times BD$, and (transposing BC^2) $AC^2 = AB^2 + BC^2 \mp 2AB \times BD$; that is, the square of the side AC subtending the acute

$\left\{\begin{matrix}\text{acute}\\ \text{obtuſe}\end{matrix}\right\}$ angle *ABC* in caſe $\left\{\begin{matrix}\text{1ſt, is leſs}\\ \text{2d, is greater}\end{matrix}\right\}$ than the ſum of the ſquares of the other two ſides by a double rectangle under the whole baſe, and the ſegments *BD*.

179. Cor. 5. When the angle *ABC* is a right angle, *BD* vaniſhes, and the ſquare of the ſide *AC* ſubtending the right angle, is exactly equal to the ſum of the ſquares of the other two ſides.

180. Scholium. This theorem and its corollaries, comprehends the 47th of the 1ſt book, and the 12th and 13th of the 2d book of the Elements of Euclid. We have here divided this 4th corollary into two caſes; one where the angle *ABC* is acute, the other where it is obtuſe: both caſes may be conſidered under one and the ſame general idea. The point *B* is conſidered as the origin of the line *BD*, if the line *BD* is conſidered as affirmative when it lies on the ſame ſide of *B* with the point *A*; then it muſt be conſidered as negative when it lies on the contrary ſide. If then $2AB \times BD$ is to be ſubtracted (by the theorem deduced as in corollary 4th) in the former caſe, it muſt be added in the latter: for now *BD* being negative, $-2AB \times -BD$ becomes $+2AB \times BD$, by the rules of algebra. Again, if *BD* is conſidered as affirmative, when it lies on the contrary ſide of *B* to the point *A*, the angle *ABC* being obtuſe; then the theorem in corollary 4th would give us $AC^2 = AB^2 + BC^2 + 2AB \times BD$; and this in the other caſe, when *BD* becomes negative (the angle *ABC* being acute) will be $AC^2 = AB^2 + BC^2 - 2AB \times BD$, as before.

181. Prop. 4. Theorem. A line drawn from the vertex of any triangle to the baſe, cuts every line parallel to the baſe, in the ſame proportion with the baſe itſelf.

Let *ABC* (fig. 23.) be the triangle, *BC* the baſe, *EF* a line parallel to the baſe, *ADM* a line drawn from the vertex *A*, cutting *EF* in *D*, and the baſe in *M*. The ſegments of the line *EF* have the ſame

proportion to each other, as the ſegments of the baſe; that is, $DE : DF :: MB : MC$.

The angles AED and ABM are equal (29. El. I.), and the angle EAD common to the two triangles AED and ABM, therefore they are equiangular. For a like reaſon, the triangles ADF and AMC are alſo equiangular; therefore

$AD : AM :: DE : MB$ (4. El. VI.): alſo
$AD : AM :: DF : MC$; therefore (11. El. V.)
$DE : MB :: DF : MC$, and alternately (16. El. V.)
$DE : DF :: MB : MC$.

Cor. If the line AM biſects the baſe, it will biſect every line parallel to the baſe.

182. Prop. 5. Theorem. In any radius OC of a circle (fig. 22.) whoſe center is O, let the point B be taken at pleaſure, and let another point A be taken in that radius produced, ſo that OA be a third proportional to OB and OC, or that OB, OC, OA be in continual proportion. I ſay, that if from the points A and B, two lines AP and BP be drawn to any point P in the circumference, then will the variable lines AP and BP always be to each other as the conſtant lines OC and OB; that is, $AP : BP :: OC : OB$ *.

Draw the radius OP, and we ſhall have two triangles BOP and AOP, having the angle at O common. But by the hypotheſis $OB : OC :: OC : OA$, or becauſe $OC = OP$, we have $OB : OP :: OP : OA$; that is, the ſides of thoſe triangles about the common angle are proportional, therefore the triangles BOP and AOP are equiangular (6. El. VI.), and their other ſides proportional (4. El. VI.); that is $AP : AO :: BP : PO$, and alternately AP is to BP as AO (or OA) to $PO = OC$, but as OA to OC, ſo is OC to OB by the hypotheſis; therefore $AP : BP :: OC : OB$ (11. El. V.).

* The lines AP and BP are called variable, becauſe they vary their length, when the place of the point P varies; but OC and OB are called conſtant, becauſe they are conſtantly the ſame, whatever be the place of the point P.

183. Cor.

183. Cor. 1. Hence the *locus* of all the points P, from whence lines drawn to two given points A and B, ſhall have a given ratio, is the circumference of a circle.

184. Cor. 2. Suppoſe the point P to remove into the point C, then AP becomes AC, and BP becomes BC, and the proportion before found, viz. $AP : BP :: CO : BO$, becomes $AC : CB :: CO : BO$. Hence if a given line AB be divided in the point C, ſo that AC is to CB in a given ratio, and it be required to find the locus of all thoſe points, from which lines drawn to the points A and B ſhall have that given ratio; then on the points C and B erect the perpendiculars Ca and Bc, and take $Ca = CA$ and $Bc = BC$, and through the points a and c draw acO, cutting AB produced in O; and O ſhall be the center, and OC the radius of the circle required. For $CO : BO :: Ca : Bc$ (4. VI. El.) or as AC to BC; therefore the point O is rightly found.

185. Cor. 3. Produce the radius CO till it again cut the circle in D, ſo that CD is a diameter; then, if we ſuppoſe the point P to remove into D, we ſhall have $AP = AD$, and $BP = BD$ and $AD : BD :: CO : BO$.

Scholium. From this general principle, That AC is to BC, or AD to BD, as CO to BO, or as AO to CO (by the hypotheſis), we may infer a variety of analogies: the moſt uſeful of which are as follows:

$$AC - BC : \begin{cases} AC :: \begin{cases} BC : CO \\ AB : AD \end{cases} \\ BC :: \begin{cases} AC : CO \\ AB : BD \\ BC : BO \end{cases} \\ AB :: \begin{cases} AC : AD \\ BC : BD. \end{cases} \end{cases}$$

To what has been here laid down we may ſubjoin the propoſitions B, C, and D in Simſon. Book VI.

186. Cor. to Prop. B. It appears in the course of the demonstration, that if a circle be circumscribed about any triangle ABC, and a chord AE be drawn so as to bisect the vertical angle, then will the rectangle under the sides of the sides of the triangle, equal the rectangle under the chord, and its segment intercepted between the vertex and the base. For it appeared, that the rectangle $BA \times AC = EA \times AD$.

THE

THE

ELEMENTS

OF

PLANE TRIGONOMETRY.

M DCC LXXXIV.

Plane Trigonometry has been treated of ſo largely, and by ſo many writers, that nothing new can be expected in the elementary part. All that is here done, is to ſelect ſuch propoſitions, as may probably be wanted in the uſual courſe of academical ſtudies; eſpecially Aſtronomy and Natural Philoſophy. The author has taken the liberty that moſt others on this ſubject have done; that of borrowing demonſtrations, and even altering them, when he thought they could be improved.

PLANE TRIGONOMETRY.

1. BY TRIGONOMETRY is ſometimes meant a part of Geometry, ſhewing the relation of certain lines, inſcribed in and circumſcribed about a circle. In this ſenſe it is to be underſtood in what now follows.

2. LEMMA 1. fig. 1. (from Dr. Simſon). Let *ACB* be a rectilinear angle: if about the point *C* as a center, and at any diſtance *CA*, a circle be deſcribed, meeting *CA* and *CB* (the ſtraight lines including the angle *ACB*) in *A* and *B*; the angle *ACB* will be to four right angles, as the arch *AB* to the whole circumference.

Produce *AC* till it meet the circle again in *D*, and through *C* draw *HL* perpendicular to *AC*, meeting the circle in *H* and *L*. By 33. VI. El. the angle *ACB* is to a right angle *ACH*, as the arch *AB* to the arch *AH* *; and quadrupling the conſequents, the angle *ACB* will be to four right angles, as the arch *AB* to four times the arch *AH*, or to the whole circumference, by 26. III. El.

3. LEMMA 2. fig. 2. (from Dr. Simſon). Let *ACB* be a plane rectilinear angle, as before; about *C* as a center, with any two diſtances *Ca* and *CA*, let two circles be deſcribed, meeting *CA* and *CB* in *a*, *b*, *A*, *B*; the arch *AB* will be to the whole circumference of which it is an arch, as the arch *ab* is to the whole circumference of which it is an arch.

By lemma 1. the arch *AB* is to the whole circumference of which it is an arch, as the angle *ACB* is to

* Any part of the circumference of a circle, as *AB* or *AH*, is called an arch.

four

four right angles; and by the ſame lemma, the arch *ab* is to the whole circumference of which it is an arch, as the angle *ACB* is to four right angles; therefore the arch *AB* is to the whole circumference of which it is an arch, as the arch *ab* is to the whole circumference of which it is an arch.

4. Cor. 1. Hence, if the circumferences of any two circles be each divided into the ſame number of equal parts, whatever number of thoſe parts is contained in any arch *AB* of one circle ſubtending a given angle *ACB*; the ſame number of parts will be contained in the arch *ab* of the other circle ſubtending the ſame angle *ACB*.

5. Cor. 2. Hence, if a circle of any radius *whatever*, be divided into 360 equal parts, called degrees; each degree into 60 equal parts, called minutes; each minute into 60 equal parts, called ſeconds, &c. the number of degrees, minutes, and ſeconds, intercepted by two radii *CA* and *CB*, will be a proper meaſure of the angle *ACB*: for the number of degrees, minutes, &c. ſo intercepted, will be to 360 degrees, as the angle *ACB* is to four right angles; therefore the number of degrees, minutes, &c. ſo intercepted, will be to the number of degrees, minutes, &c. intercepted by any other two radii, as the magnitude of the former angle to the magnitude of the latter. And this is the proper meaning of a *meaſure*. One quantity is ſaid to be a *meaſure* of another, when the meaſure in one caſe is to the meaſure in any other, as the magnitude of the quantity to be meaſured in the former caſe is to the magnitude of that quantity in the latter.

6. The whole circle then conſiſting of 360 degrees; a right angle will be 90 degrees, two right angles will be 180 degrees, half a right angle 45 degrees, each of the angles of an equilateral triangle will be 60 degrees.

The magnitude or quantity of any angle is noted thus; $\overset{D}{21} : \overset{M}{16} : \overset{S}{45}$, or thus, $21^{\circ} : 16' : 45''$.

DEFI-

DEFINITIONS.

7. With the center *C* (fig. 4.) and radius *CA* deſcribe a circle; produce *AC* till it meet the circle again in *D*, ſo that *AD* may be a diameter. Draw *HL* a diameter perpendicular to *AC*. Theſe two diameters will divide the circumference into four equal arches, called quadrants, each arch containing 90 degrees (by the laſt); draw *tAT* touching the circle in *A*: laſtly, draw the radius *CB*, and produce it till it meet the line *tAT* in *T*.

8. DEF. 1. The difference of any arch from a quadrant, or of any angle from 90 degrees, is called the *complement* of that arch or angle. The like difference from a ſemicircle, or 180 degrees, is called the *ſupplement*. Thus, let the arch *AB* be reckoned from *A as its beginning* *, then *HB* is the complement of *AB*; the angle *HCB* the complement of *ACB*; alſo the arch *DHB* is the ſupplement of *AB*, and the angle *DCB* the ſupplement of *ACB*.

9. DEF. 2. The *chord* of any arch, is a right line drawn from one extremity of the arch to the other. Let the points *B* and *E* be in the circle; join *B* and *E*, and the right line *BE* is the chord of the arch *BAE*, or of the angle *BCE* of which that arch is the meaſure.

10 DEF. 3. The *ſine* or *right ſine* of any arch, is a right line drawn from the end of the arch perpendicular to a diameter, paſſing through the beginning of the arch. From *B* let down *BF* perpendicular to *AD*, and *BF* is the ſine of the arch *AB*, or of the angle *ACB*. Again: let *ABHb* be the ſupplement of *AB*; draw the radius *Cb*, and produce it till it meets the circle in *E*, and the line *TAt* in *t*; laſtly, let down *bf* and *EF* perpendicular to the diameter *AD*, and *bf*

* A geometrical circle has no beginning. But in the application of trigonometry, to moſt ſubjects (aſtronomy eſpecially) it is neceſſary to fix upon ſome point as a beginning, from whence all arches are to be computed. Thus, the degrees on the equator and ecliptic are reckoned from one of the equinoctial points.

is the ſine of the arch *ABHb*, or of the angle *ACb*, and *EF* is the ſine of the arch *ABbDLE*, or of the angle which it meaſures.

11. Def. 4. The *verſed ſine*, is that part of the diameter paſſing through the beginning of the arch, which is intercepted between the beginning of the arch and the right ſine: thus, *AF* is the verſed ſine of the arch *AB*, or of the angle *ACB*; and *Af* is the verſed ſine of the arch *AHb*, or of the angle *ACb*.

12. Def. 5. The *tangent* of an arch, is a right line touching the circle in the beginning of the arch, produced from thence till it meets the radius (produced) drawn through the end of the arch. Thus, *AT* is the tangent of the arch *AB*, or of the angle *ACB*; and *At* is the tangent of the arch *AHb*, or of the angle *ACb*.

13. Def. 6. The *ſecant* of an arch, is a right line drawn from the center through the end of the arch, and produced till it meets the tangent. Thus *CT* is the ſecant of the arch *AB*, or of the angle *ACB*; and *Ct* is the ſecant of the arch *AHb*, or of the angle *ACb*.

14. Def. 7. The *co-ſine* of an arch, is the part of the diameter paſſing through the beginning of the arch, which is intercepted between the center and right ſine. Thus, *CF* is the co-ſine of the arch *AB*, or of the angle *ACB*; and *Cf* is the co-ſine of the arch *AHb*, or of the angle *ACb*.

15. Def. 8. The *co-tangent* of an arch, is a line touching the circle in the end of the firſt quadrant, produced from thence till it meets the radius (produced) drawn through the end of the arch. Thus, draw *HK*, touching the circle in *H*, meeting the radius *CB* produced in *K*, and *HK* is the co-tangent of the arch *AB*, or of the angle *ACB*.

16. Def. 9. The *co-ſecant* of an arch, is a right line drawn from the center through the end of the arch, and produced till it meets the co-tangent. Thus, *CK* is the co-ſecant of the arch *AB*, or of the angle *ACB*.

17. Scho-

17. SCHOLIUM to definitions 7, 8, 9. It is manifest, that the co-tangent and co-secant are referred to the diameter *HL*, passing through the end of the quadrant, in like manner as the tangent and secant are to the diameter *AD*, passing through the beginning of the quadrant.

18. It appears also, that the co-sine, co-tangent, and co-secant of an arch under 90 degrees, or of an angle less than a right angle, are respectively equal to the sine, tangent, and secant of the complement of that arch or angle: for, drawing *BI* perpendicular to the diameter *HL*, we have $CF=BI$; but *BI*, *HK*, *CK* are respectively the sine, tangent, and secant of the arch *HB*, reckoned from *H*, as its beginning. But the arch *HB* is the complement of the arch *AB*; and the angle *HCB* is the complement of the angle *ACB*: therefore, the co-sine, co-tangent, and co-secant of any arch or angle, are equal to the sine, tangent, and secant of the complement of that arch or angle, whence they have their names.

19. Moreover: the sine, co-sine, tangent, secant, &c. of any angle *ACB*, in a circle whose radius is *AC* (fig. 4.) will be to the sine, co-sine, &c. of the same angle *ACB*, in a circle whose radius is *AC* (fig. 3.) respectively, as the radius of the former circle to the radius of the latter circle: for the several right angled triangles corresponding to each other in fig. 4. and fig. 3. having one acute angle in each equal (viz. *ACB* in fig. 4. equal to *ACB* in fig. 3.) are equiangular; therefore *BF* (fig. 4.) : *BF* (fig. 3.) :: *BC* (fig. 4.) : *BC* (fig. 3.), and so in the case of all the other corresponding triangles, in each of which one corresponding side is radius.

20. Hence, if the radius of any circle be divided into 10,000,000 equal parts, and the length of the sine, tangent or secant, &c. of any angle in such parts be given; the length of the sine, tangent, and secant of the same angle to any other given radius may be found. A table exhibiting the length of the sine,

 tan-

tangent, and ſecant of every degree and minute of the firſt quadrant in ſuch parts, whereof 10,000,000 make the radius, is called a trigonometrical canon; and it will always be, as the tabular radius is to any other given radius, ſo is the tabular ſine, &c. of any angle to the ſine, &c. of the ſame angle to the given radius.

21. COROLLARIES to the definitions.

1. The chord of 60 degrees is radius: for then the triangle *ACB* is equiangular and equilateral.

2. The ſine of 90 degrees is equal to radius.

3. The tangent of 45 degrees is radius: for then the angle *ACT* being half a right angle, the other acute angle *ATC* muſt be ſo too; the triangle *ACT* iſoſceles, and $AC = AT$.

4. The ſecant of 0 (or the beginning of the circle) is radius.

5. The co-ſine of no degrees is radius, the co-ſine of 90 degrees is nothing.

6. The verſed ſine of 90 degrees is radius; the verſed ſine of 180 degrees is the diameter.

7. Univerſally; the verſed ſine is always either the ſum or the difference of the co-ſine and radius, viz. their ſum in the two middle quadrants *HD* and *DL*, and their difference in the two extreme quadrants *AH* and *LA*.

22. SCHOLIUM. The ſeveral changes in the algebraic ſigns of all theſe lines, muſt be particularly obſerved in the application of trigonometry to aſtronomical and phyſical problems*. They alſo ſerve much to illuſtrate the uſe of the negative ſign in the application of algebra to geometry. We ſhall therefore trace out all theſe changes particularly.

23. And firſt; the ſine increaſes from 0 during the firſt quadrant *AH*, when it becomes radius. After that it decreaſes during the ſecond quadrant *HD*, till it again becomes nothing. After this, the ſines will lie on the contrary ſide of the diameter *AD*, from

* See a remarkable inſtance in Newton's *Principia*, p. 440. ed. 3.

whence

whence their length is computed; therefore being reckoned affirmative before, muſt be now reckoned negative. During the third quadrant *DL*, this negative ſine *βf* increaſes till it becomes equal to radius (but is negative). During the laſt quadrant *LA*, it decreaſes till it becomes nothing, when the arch is 360 degrees; after which it is affirmative, and increaſes as before.

24. The co-ſine, when the arch is 0, is equal to radius, but decreaſes for the firſt quadrant *AH*, at the end of which it is nothing; after this it is negative. For its length being computed from the center *C*, the co-ſine *Cf* will lie in an oppoſite direction to the co-ſine *CF*. This negative co-ſine increaſes during the ſecond quadrant *HD*, at the end of which it is equal to radius (but is negative). It decreaſes for the third quadrant *DL*, at the end of which it is nothing. In the fourth quadrant *LA*, it is again affirmative, and increaſes till it is again equal to radius, as before.

25. The tangent at the beginning is nothing, and increaſes to *infinity* during the firſt quadrant *AH*; that is, there is no line that can be aſſigned, how great ſoever its length, but you may find an angle under 90 degrees, whoſe tangent ſhall exceed that line, ſo that the tangent has no limit to its increaſe, as the ſine has. In the ſecond quadrant *HD*, the tangent *At* is negative: for the tangents being computed from the point *A*, the tangent *At* will lie in a direction oppoſite to *AT*. During this quadrant, it decreaſes from an infinite negative to nothing. In the third quadrant *DL*, it is again affirmative, and increaſes from nothing to infinity, juſt as in the firſt quadrant. In the fourth quadrant *LA*, it decreaſes from an infinite negative to nothing, as in the ſecond quadrant; after which it is affirmative, and increaſes as before.

26. The ſecant at the beginning of the firſt quadrant (when the arch is nothing) is equal to radius; it increaſes for the firſt quadrant, at the end of which

it is infinite (in the ſenſe before explained). In the ſecond quadrant *HD*, the ſecant is negative: for the ſecant has its origin at the center *C*, and its length is computed from that point, to its concourſe with the tangent. In the firſt quadrant this length is reckoned from *C* towards *B* and *T*; but in the ſecond quadrant it is reckoned (on the revolving radius) in a contrary direction, from *C* towards *E* to *t*, and is therefore negative. During the ſecond quadrant, this negative ſecant decreaſes from infinity till it is equal to radius. In the third quadrant *DL*, it increaſes again from radius to infinity, but continues all this time negative; for the interſection which this revolving radius (produced) makes with the tangent, continues on the ſame ſide (on that revolving line) with reſpect to the point *C*, both for the ſecond and third quadrants: but in the fourth quadrant *LA*, this interſection changes to the oppoſite part of the revolving radius (produced) namely, the ſame as at firſt; therefore in the fourth quadrant the ſecant is affirmative, and decreaſes from infinity till it becomes radius as at firſt. Thus the ſecant has the ſame algebraic ſign with the co-ſine.

27. We may obſerve here, that the ſine and co-ſine never exceed radius; the ſecant and co-ſecant are never leſs than radius: but the tangent admits of all poſſible degrees of magnitude.

28. Moreover; all theſe lines change their direction as often as they become either infinite or nothing. When they become infinite, their increaſe is at its utmoſt limit, and after this they change their direction and alſo decreaſe. When they become nothing, their decreaſe is at its utmoſt limit, and they then increaſe again in a contrary direction. Thus theſe quantities change their algebraic ſigns, when (as ſome expreſs it) they paſs through a ſtate of *infinity* or a ſtate of *nothingneſs*.

29. In like manner we may trace all the changes, both in the direction and in the length of the co-tangent *HK*, and of the co-ſecant *CK*, by conſidering that

that they are computed from the points H and C, in the ſame manner as the tangent and ſecant are computed from the points A and C. The co-tangent changing its direction every quadrant, changes its ſign every quadrant, and therefore has always the ſame algebraic ſign with the tangent. The co-ſecant begins from infinity, and is affirmative for the two firſt quadrants, and negative for the two laſt, having the ſame algebraic ſign with the ſine.

30. The verſed ſine increaſes from nothing during the two firſt quadrants, till it becomes the diameter, which is its utmoſt limit. It then decreaſes for the two laſt quadrants, till it becomes nothing: but being always computed in the ſame direction (from A towards D) is always affirmative.

31. The changes of the algebraic ſigns of theſe ſeveral lines may be ſeen in one view, in this table.

		Sine.	Co-ſine.	Tan.	Co-tan.	Sec.	Co-ſec.
[1]st, 5th, 9th	quadrants	+	+	+	+	+	+
2d, 6th, 10th	quadrants	+	−	−	−	−	+
3d. 7th, 11th	quadrants	−	−	+	+	−	−
4th, 8th, 12th	quadrants	−	+	−	−	+	−

32. Prop. 1. fig. 4. Let the arches Ab and AB, be ſupplements to each other, viz. Ab greater, and AB leſs than a quadrant, then will their ſines bf and BF be equal.

For, becauſe $Ab+AB=180^{D}=Ab+bD$, therefore $AB=bD$, and the angle $bCD=BCA$; but $bC=BC$, therefore the right angled triangles bCf and BCF are equal, and $bf=BF$, by 26 I. El.

In like manner, Cf the co-ſine of the arch AHB is equal to CF the co-ſine of the arch AB; only as it falls on the other ſide of the point C, from whence the co-ſines have their origin, it will be negative.

Again, At the tangent, and Ct the ſecant of Ab are reſpectively equal to AT the tangent, and CT the ſecant of AB: for, becauſe BCA or $TCA=bCD=tCA$, and CA is common to the two right angled triangles

TCA and *tCA*, they are equal (26. I. El.) and $At = AT$ and $Ct = CT$, only the tangent and ſecant being now produced in a contrary direction will be negative.

In like manner the ſine, co-ſine, tangent, and ſecant of any arch, terminating in the third quadrant *DL*, will be the ſame with that of an arch equal to the exceſs of the propoſed arch above a ſemi-circle. Thus, the ſine of the arch $AHD\beta$ is βf, and is equal to *BF* the ſine of the arch $AB = D\beta$.

The ſine, co-ſine, tangent and ſecant of an arch terminating in the fourth quadrant *LA*, will be the ſame with that of an arch equal to the ſupplement of the propoſed arch to a whole circle. Thus, the ſine of the arch *AHDLE* is *EF*, and is equal to *BF*, the ſine of the arch $AB = AE$, the ſupplement of *AHDLE* to a whole circle.

The verſed ſine *Af* of an arch *Ab* above one, but under two quadrants, is equal to the difference between the verſed ſine of its ſupplement and the diameter; that is, $Af = AD - AF$: for, becauſe $CF = Cf$, therefore $Af = DF = AD - AF$.

The verſed ſine of an arch above two quadrants is (not merely equal, but) the ſame with the verſed ſine of its ſupplement to a whole circle. Thus, the verſed ſine of the arch $AHD\beta$ and alſo of the arch $AEL\beta$ (or rather its equal *ABHb*) is *Af*. And the verſed ſine of the arch *AHDLE*, and of the arch $AB = AE$, is in both caſes *AF*.

From the foregoing propoſitions it follows, that a table of ſines, tangents, ſecants and verſed ſines, made for every degree and minute of the firſt quadrant, will ſerve for the whole circle.

32. Prop. 2. fig. 4. The right angled triangles *BCF*, *TCA*, *CKH*, having the ſeveral acute angles *BCF*, *TCA*, *CKH* (29. I. El.) equal, are equiangular; whence we have the following analogies:

1st	*CF*	*BF*	*CA*	*TA*
	Co-ſine.	Sine.	Radius.	Tangent.

2d

2d	*CF* Co-fine.	*CB* Radius.	*CA=CB* Radius.	*CT* Secant.
3d	*BF* Sine.	*CB* Radius.	*CH=CB* Radius.	*CK* Co-fecant.
4th	*BF* Sine.	*CF* Co-fine.	*CH* Radius.	*HK* Co-tangent.
5th	*TA* Tangent.	*CA* Radius.	*CH=CA* Radius.	*HK* Co-tangent.

Hence the radius is a mean proportional between the co-fine and fecant, or between the fine and co-fecant, or between the tangent and co-tangent.

33. Prop. 3. fig. 4. The fine of any arch is equal to half the chord of double the arch.

Let *BF* be the fine of the arch *AB*, produce *BF* till it meet the circle again in *E*, and *BF* = half *BE* (3. III. El.). But *BCF* = *ECF* (8. I. El.) therefore the arch *AB* is equal to the arch *AE* (26. III. El.) and the arch *BAE* is double the arch *AB*; but *BE* is the chord of the arch *BAE*, therefore the fine of any arch is half the chord of double that arch.

Cor. The fine of 30 degrees is half the radius.

Converfely. The chord of any arch is double the fine of half that arch; for, let *BE* be the chord of the arch *BAE*, draw the radius *CA* perpendicular to *BE*, and cutting it in *F*; then *BF* is the right fine of the arch *BA*: but *BE* is double of *BF* (3. III. El.) and the arch *BA* is half the arch *BE*, by 8. I. El. and 26 III. El. as before; therefore the chord is double the fine of half the arch.

34. Prop. 4. fig. 5. On the diameter *AD* defcribe a femicircle *ABD*; draw *AB* the chord and *BF* the fine of the arch *AB*, draw the radius *CLM* perpendicular to the chord in *L*, and cutting the circle in *M*; then will this radius *CLM* bifect the chord *AB* in *L* (3. III. El.) and the arch *AMB* in *M*, as in par. 33. Moreover *AL* will be the fine of the arch *AM*, and *CL* its co-fine. Laftly, join *DB*, and the triangle *ADB* will be right angled at *B* (31. III. El.) and will be divided

by

by BF, a perpendicular drawn from the right angle to the baſe, into two triangles AFB and DFB, ſimilar to the whole and to each other (8. VI. El.). Alſo, the right angled triangles ALC and ABD, having the acute angle LAC common, are ſimilar to one another and to the two triangles AFB and DFB. This premiſed, the triangles DAB and BAF give the following analogy; $DA : BA :: BA : AF$; that is, the chord of any arch is a mean proportional between the diameter and verſed ſine of that arch.

+

Hence $DA : AF :: DA^2 : BA^2 :: CA^2 : AL^2$; that is, the diameter is to the verſed ſine of any arch, as the ſquare of radius to the ſquare of the right ſine of half that arch.

From the triangles CAL and BAF, we have $CA : AL :: AB$ or $2AL : AF$; that is, radius is to the ſine of any arch AM, as twice that ſine to the verſed ſine of AMB double that arch.

Hence the ſquare of the right ſine of any arch is *as* the verſed ſine of double that arch.

From the ſame triangles we have $CA : CL :: AB$ or $2AL : BF$; that is, radius is to the co-ſine of any arch AM, as twice the ſine of that arch is to the ſine of AMB double that arch.

Hence the rectangle under the ſine and co-ſine of any arch, is *as* the ſine of double that arch.

The application of trigonometry to the meaſuring the ſides and angles of triangles deſcribed on a plane, or PLANE TRIGONOMETRY properly ſo called.

35. PROP. 5. In any right angled triangle ABC (fig. 6.) if the hypothenuſe AC be made radius, the perpendicular BC becomes the ſine, and the baſe AB the co-ſine of the angle at the baſe CAB. If the baſe AB (fig. 7.) be made radius, the perpendicular BC becomes the tangent, and the hypothenuſe AC the ſecant of the angle at the baſe CAB. This is manifeſt from the definitions.

COR. Hence by par. 20. in any right angled triangle, if

if one ſide and an acute angle, or two ſides only be given, the remaining ſides and angles may be found.

36. PROP. 6. The ſides of any triangle are to one another as the ſines of their oppoſite angles. Thus, in the triangle *ABC* (fig. 8. and 9.) the ſide *AB* is to the ſide *AC*, as the ſine of the angle *ACB*, oppoſite to the former ſide *AB*, is to the ſine of the angle *ABC*, oppoſite to the latter ſide *AC*.

DEMONSTRATION from Dr. Simſon (fig. 8. and 9.) From *C* and *B* draw *CD* and *BE* perpendicular to the oppoſite ſides *AB* and *AC* (produced if need be, fig. 9.). Then if *C* be made the center, and *BC* the radius of the circle, *BE* will be the ſine of the angle *ACB*; but if *B* be made the center, and the ſame *BC* the radius, *CD* will be the ſine of the angle *ABC*. Now, the right angled triangles *ADC* and *AEB*, having the acute angle at *A* common, are equiangular; therefore $AB : AC :: BE : CD$, or *AB* to *AC* at the ſine of the angle *ACB* to the ſine of the angle *ABC*.

DEMONSTRATION from Mr. T. Simpſon (fig. 10.) On the ſide *BA* (produced if need be) take $BE = CA$, and let fall the perpendiculars *AD* and *EF*, which will be the ſines of the angles *ACB* and *ABC* to the equal radii *CA* and *BE*: but the right angled triangles *ADB* and *EFB*, having the acute angle at *B* common, are equiangular; therefore $AB : EB$ or $AC :: AD : EF$, that is, *AB* to *AC*, as the ſine of the angle *ACB* to the ſine of the angle *ABC*.

COR. 1. Hence in any triangle, if a ſide and angle oppoſite to each other be given, and alſo one other ſide, or one other angle, the remaining ſides and angles may be found.

COR. 2. Hence, if all the angles of any triangle be given, the proportion of the ſides to one another may be found.

37. LEMMA. If the ſemi-ſum and ſemi-difference of any two quantities be added together, the aggregate will be the greater quantity: and if the ſemi-difference be ſubtracted from the ſemi-ſum, the remainder

will be the less quantity. This is demonstrated by all the writers on algebra. See Introduction to Algebra, par. 217.

Cor. If the semi-sum be subtracted from the greater quantity, the remainder will be the semi-difference.

38. Prop. 7. In any triangle, where the perpendicular from the vertex falls within the base, the base will be to the sum of the sides, as the difference of the sides is to the difference of the segments of the base made by the perpendicular.

Let ABC (fig. 11.) be the proposed triangle, C the vertex, AB the base, CD the perpendicular, dividing the base into the segments AD and BD. With the center C and radius CB, the less of the two sides, describe a circle, cutting the base in G, the side AC in F, and that side produced in H; then will AH be the sum of the sides AC and CB, and AF their difference. Again (because DB and DG are equal, by 3. III. El.) AG will be the difference of the segments AD and DB; but by 37. III. El. $AB \times AG = AH \times AF$; therefore $AB : AH :: AF : AG$, 16. VI. El.; that is, the base is to the sum of the sides, as the difference of the sides to the difference of the segments of the base.

Cor In any triangle ABC, if a perpendicular be let fall from the greatest angle, it will always fall within the opposite side or base, and will divide the triangle into two right angled triangles ACD and BCD, whose hypothenuses, AC and BC, are the sides of that triangle, and whose bases, AD and BD, are the segments of the base of that triangle. Now, if all the three sides of the triangle ABC be given, then the base AB, or sum of the segments AD, DB is given, and their difference may be found by this proposition; but if their sum and difference be known, the segments themselves may be found by the lemma. Thus, in each of the two right angled triangles ACD and BCD, we have the hypothenuse (AC and BC) and the base (AD and BD), whence the angles at the base CAD and CBD may be found by prop. 5.; consequently

quently all the angles of the triangle *ABC* are known by cor. 32. I. El.

39. Prop. 8. In any triangle, the ſum of any two ſides is to their difference, as the tangent of the ſemi-ſum of the angles at the baſe to the tangent of their ſemi-difference.

Let *ABC* (fig. 12.) be the propoſed triangle, whoſe ſides are *AC* and *BC*, and baſe *AB*, with the radius *CB* (the leſs of the two ſides) and center *C*, deſcribe a circle cutting the longeſt ſide *AC* in *F*, that ſide produced in *H*; then is *AH* the ſum of the ſides *AC* and *CB*, and *AF* their difference. Join *FB* and *HB*, then is the angle *HCB* the ſum of the angles at the baſe (by 32. I. El.), and *HFB* is half the angle *HCB* (by 20. III. El.), or the ſemi-ſum of the angles at the baſe. Now the triangle *FCB* is iſoſceles, and $CFB = CBF$ (by 5. I. El.), therefore *CBF* is alſo the ſemi-ſum of the angles at the baſe: but $ABF = CBA - CBF$; that is, *ABF* is the difference between the greater of the angles at the baſe, and their ſemi-ſum, therefore, by cor. to lemma, par. 37. *ABF* is the ſemi-difference of the angles at the baſe.

Laſtly, draw *FE* parallel to *BH*, and the angle $EFB = HBF$ (by 29. I. El.); that is, equal to a right angle (by 31. III. El.): therefore if *F* be made the center and *FB* the radius, *HB* will be the tangent of *HFB*, the ſemi-ſum of the angles at the baſe; and if *B* be made the center, and the ſame *FB* the radius, then *FE* will be the tangent of *EBF* the ſemi-difference of the angles at the baſe. But the triangles *AFE*, *AHB*, having the angles *AFE* and *AHB* equal (29. I. El.) and the angle at *A* common, are ſimilar; therefore $AH : AF :: HB : EF$; that is, the ſum of the ſides is to the difference of the ſides, as the tangent of the ſemi-ſum of the angles at the baſe to the tangent of their ſemi-difference.

Cor. Hence, if the ſides *AC* and *BC* of any triangle, and the angle *ACB* included between thoſe ſides, be given, the reſt of the ſides and angles may be found.

found. For, from the included angle *ACB*, we get its supplement to 180 degrees, equal to *HCB*; half of which is the semi-sum of the angles at the base, and their semi-difference will be found by this proposition. Having the semi-sum and semi-difference of the angles at the base, the angles themselves, *CBA* and *CAB*, may be found by the lemma, par. 37. Having thus got all three angles and two sides, the remaining side may be found by prop. 6.

PROBLEMS for the Exercise of Learners.

Note. Before the learner sets about the solution of these problems, he should make himself acquainted with decimal fractions, and the practical use of the tables of logarithms, sines and tangents; all which is briefly taught in the introduction usually prefixed to those tables.

1. The height of a certain tower is observed to be 65 degrees, and the distance from the foot of the tower to the place of observation was measured 80 feet: What is the height of the tower *?

2. It was observed, that a ladder 40 feet long would just reach the top of a building, when the foot of the ladder was 15 feet from the bottom of it: What

* The word *height* may be taken in two senses, it may either mean an angle, and then its quantity is given in degrees, &c.; or it may mean a linear measure, and then it is given in yards or feet, &c.

When the height of an object is given in degrees, &c. it means the angle which a line drawn from the summit of the object to the eye, makes with the horizontal plane, on which both the object and observer are supposed to stand. When the height of the sun or a star is given in degrees, it means the angle which a line drawn from the center of the sun, or star, to the eye of the observer, makes with the horizon, or with the horizontal plane on which the observer stands. But height also means the linear measure of any column, &c. standing perpendicularly on an horizontal plane. This is properly the *perpendicular* height; but the word *height* is here used without any distinction; it is left to the learner to find that out from the nature of the question.

is

is the height of that building? and, What is the inclination of the ladder to the horizon?

3. What muſt be the length of a ſcaling ladder to ſcale a tower, whoſe height is $67^d : 41^m$, acroſs a ditch 16 feet broad?

4. A tower, whoſe height is known to be 100 feet, ſubtends an angle of 25 degrees (with the horizon): What is the diſtance of the obſerver?

5. What angle will that tower ſubtend, when the obſerver is twice as far from the tower? and, What angle will it ſubtend, when he is only half as far?

6. What is the height of the ſun, when a man's ſhadow is half his height? and, What is the height of the ſun, when a man's ſhadow is double his own height?

7. There are three towns *A*, *B* and *C*, at *A* the towns *B* and *C* make an angle of $20^d : 12^m$; at *B*, the towns *A* and *C* make an angle of $38^d : 30^m$; laſtly, from *A* to *B* meaſures two miles and three quarters: What is the diſtance of *C* from *A*, and from B?

8. A man travels from *A* to *B*, three miles and $\frac{7}{10}$ths: then, *bending a little to the right*, he goes from *B* to *C*, which is four miles and $\frac{7}{10}$ths; at *C* he obſerves that *A* and *B* now make an angle of $29^d : 16^m$: What is his diſtance from home by the ſhorteſt cut?

9. A man travels from *A* to *B*, three miles and $\frac{7}{10}$ths; *returning back* in a miſt he loſes his way, and going a little too much *on the right hand* comes to *C*, which is four miles and $\frac{7}{10}$ths from *B*. It now clearing up, he could ſee both *A* and *B*, and obſerves that they now make an angle of $29^d : 16^m$: What is now his diſtance from home*?

10. There are three towns *A*, *B* and *C*; from *A* to *C* is three miles and three furlongs; from *B* to *C* is

* This problem at firſt ſight ſeems not to differ from the laſt. The anſwer to the former problem is, that his diſtance from home is 7 miles; the anſwer to the latter is, that his diſtance from home is 1 mile and $\frac{2}{10}$ths of a mile. It is left to the learner to make this out.

four

four miles and five furlongs *. Between *A* and *B* lies a large wood, which prevents theſe towns from being ſeen from each other, or their diſtance from being meaſured. However, *A* and *B* are both viſible from *C*, and there make an angle of $71^d : 2^m$. *How* † muſt I cut a viſta through that wood, ſo that the towns *A* and *C* may be ſeen from each other, and what will be their diſtance?

11. There are three towns *A*, *B* and *C*: the town *A* is diſtant from *B* five miles, *B* is diſtant from *C* ſeven miles, and *C* is diſtant from *A* nine miles: What are their reſpective *bearings* from each other?

And ſuppoſe their reſpective diſtances had been one mile, three miles, and five miles: What would have been their bearings in that caſe?

12. How may I plant three trees, ſo that the angles they ſhall make with each other may be 50, 60 and 70 degrees?

13. How muſt three trees, *A*, *B* and *C*, be planted, ſo that the angle at *A* may be double the angle at *B*, and the angle at *B* may be double the angle at *C*, and ſo that a line of 100 yards may juſt go round them?

14. The height of a certain tower was obſerved to be 20 degrees. The obſerver intended to meaſure his diſtance from the foot of the tower to obtain its height, as in problem 1. But when he had meaſured out 85 feet in a direct line towards the tower, he was ſtopped by a ditch: however, the height of the tower was again obſerved, and found to be $31^d : 34^m$: What is the height of the tower? and, How much did the obſerver want of getting to the tower, when he was ſtopped by the ditch?

15. On the 21ſt of May, between 8 and 9 in the morning, the ſun being then on the E S E (eaſt ſouth eaſt) point of the compaſs, and his altitude $38^d : 49^m$, a ſmall cloud was obſerved in the S W (ſouth weſt),

* Note, Eight furlongs make a mile.

† It is left to the learned to find out the import of this word *How*?

whoſe

whose shadow fell on a spot of ground due west from the place of observation, and whose distance was known to be just one mile and an half: What was the height of that cloud above the surface of the earth, and over *what place* was it perpendicular *?

16. There is a certain mountain, the perpendicular height of whose summit above the neighbouring plane is required. To determine this, two stations were found upon the plane in sight of the highest cliff, and in sight of each other. At the first station, *A* the height of the cliff was observed to be $22^d : 49^m$, and the angle between the cliff and second station *B*, was found to be $39^d : 10^m$. At the second station *B*, the angle between the cliff and first station *A*, was observed to be $48^d : 45^m$. Lastly, the distance between the two stations was measured, and found to be 240 yards. What is the height of the cliff above the plain? and, What part of the base line (or that line produced) is the nearest to the cliff?

17. In order to complete some chorographical observations about Cambridge, it was necessary to measure exactly the distance between the observatory at Trinity college and St. Mary's steeple; but this could not be done by direct measurement, by reason of the houses between them; therefore recourse was had to other methods. Two stations were chosen in the fields behind Trinity college, from each of which, the observatory, St. Mary's, and the other station, could be all three seen at once. At the first station, the angle made by the top of the observatory and the top of St. Mary's steeple, was $14^d : 34^m$. The angle between the top of the observatory and second station, was found to be $60^d : 50^m$. The angle between the top of St. Mary's steeple and the second station, was found to be $46^d : 16^m$. These observations at the first station being compleated, the instruments were removed to

* Note, the compass consists of 32 points. The ESE is two of those points distant from the East towards the South. The SW is four of those points distant from the South towards the West.

the ſecond ſtation. At the ſecond ſtation, the angle between the top of the obſervatory and the firſt ſtation, was found to be $96^{d} : 44^{m}$. The angle between the top of St. Mary's and the firſt ſtation, was found to be $115^{d} : 23^{m}$. Laſtly the baſe line or diſtance between the two ſtations, was exactly meaſured and found to be 908,36 feet.

What is the horizontal diſtance between the obſervatory and St. Mary's ſteeple from theſe *data?*

Firſt, Suppoſing the two buildings of the ſame height? Anſ. 674,62 feet.

Secondly, Suppoſing the obſervatory to be 73,50 feet, and St. Mary's ſteeple 97,25 feet high? Anſ. 674,20 feet. Hence the error in the computed diſtance of theſe two buildings, ariſing from a neglect of the inequality of their heights, is leſs than half a foot, and is altogether inconſiderable.

N. B. All the meaſures here laid down, were taken with great exactneſs, by the late Dr. Maſon, Fellow of Trinity College, Cambridge.

FINIS.

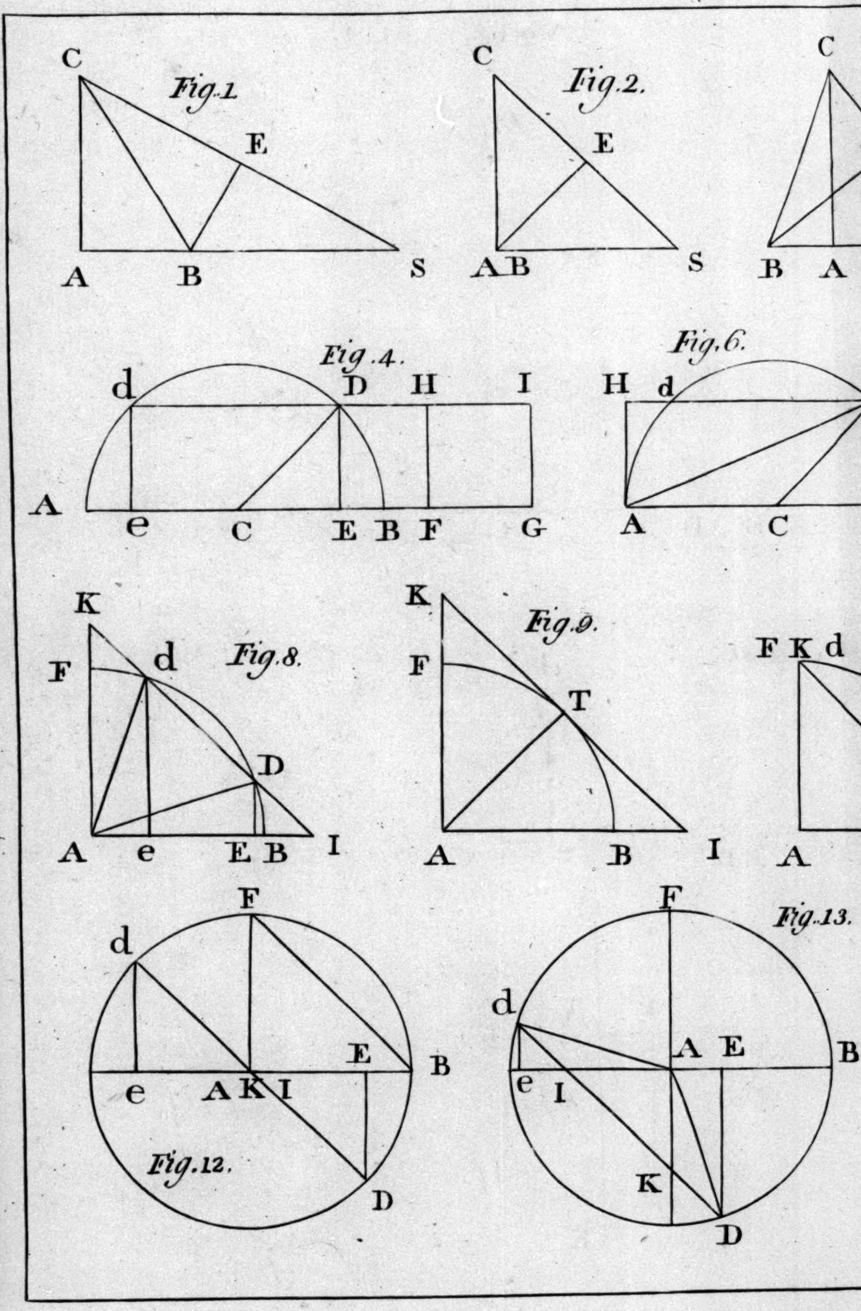

Fig. 1.
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Fig. 2.
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Fig. 4.
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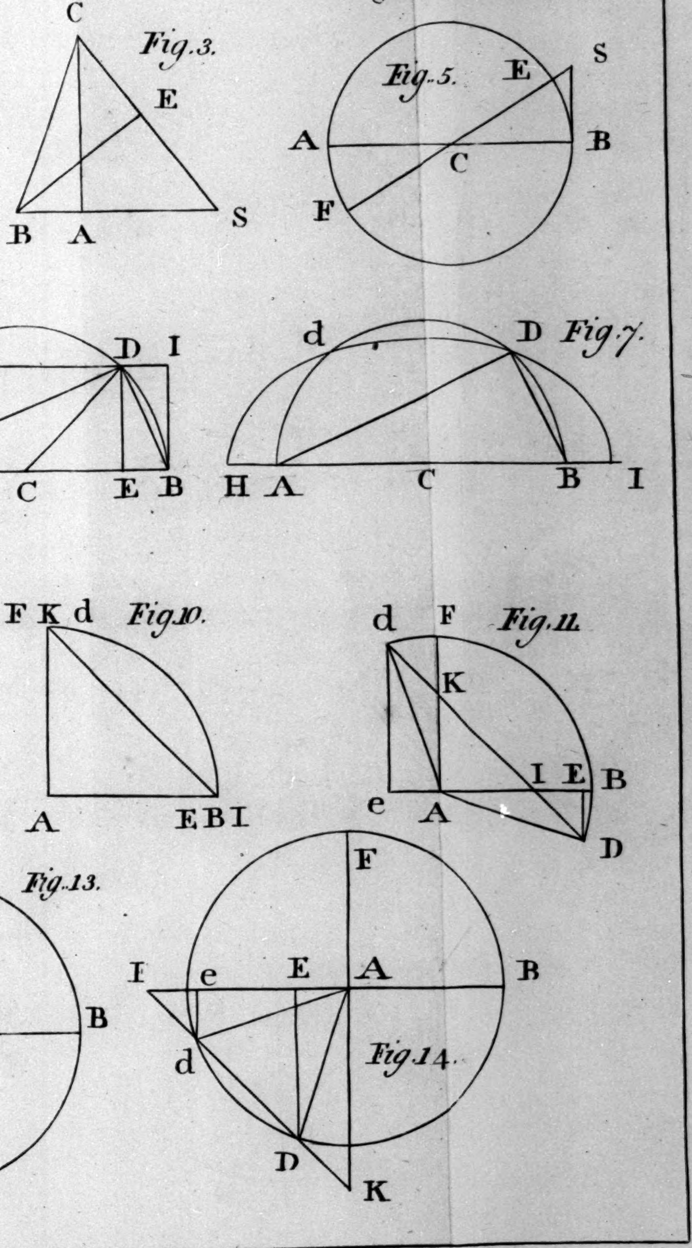
Algebra Tab. 1.



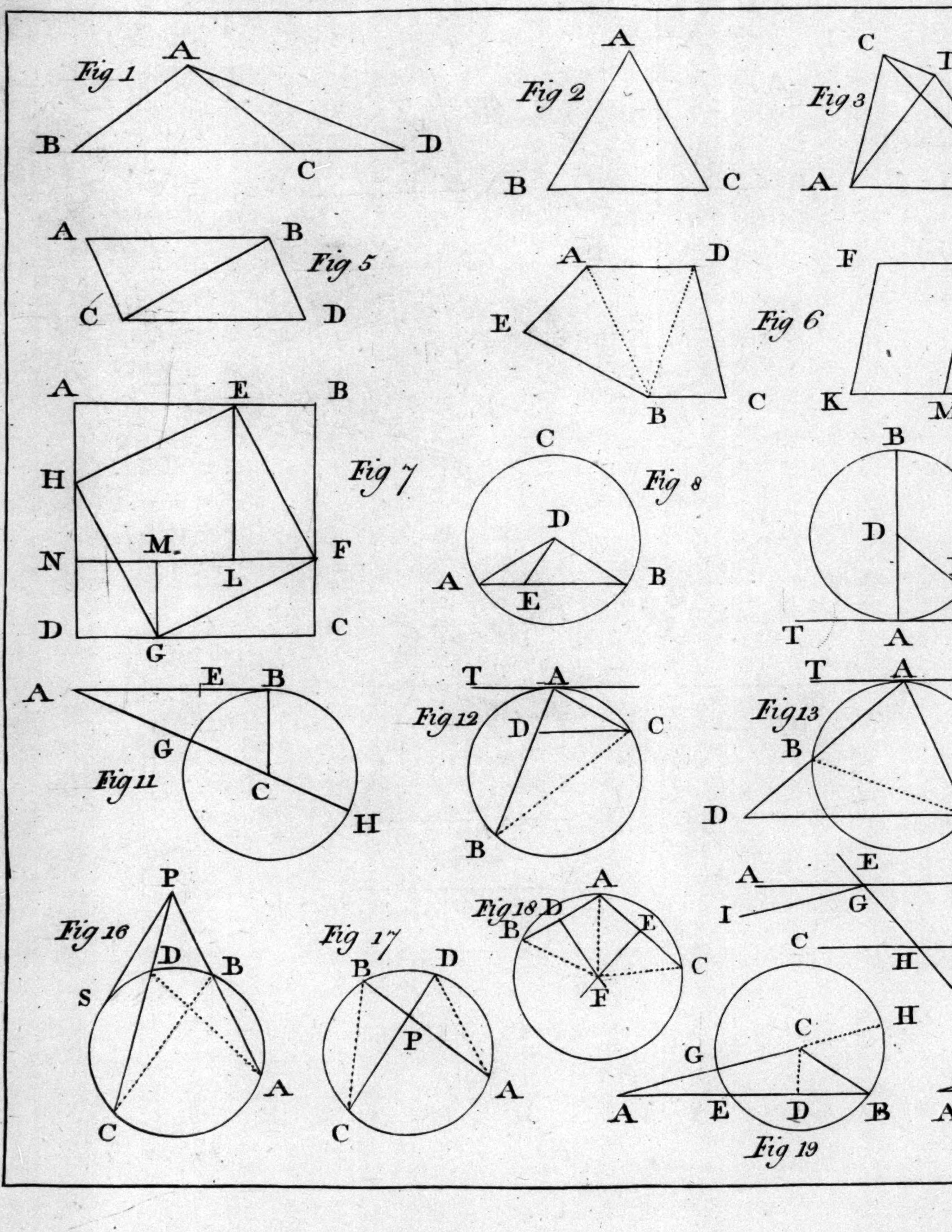


Fig 1
Fig 2
Fig 3
Fig 5
Fig 6
Fig 7
Fig 8
Fig 11
Fig 12
Fig 13
Fig 16
Fig 17
Fig 18
Fig 19

Geometry Tab 11.

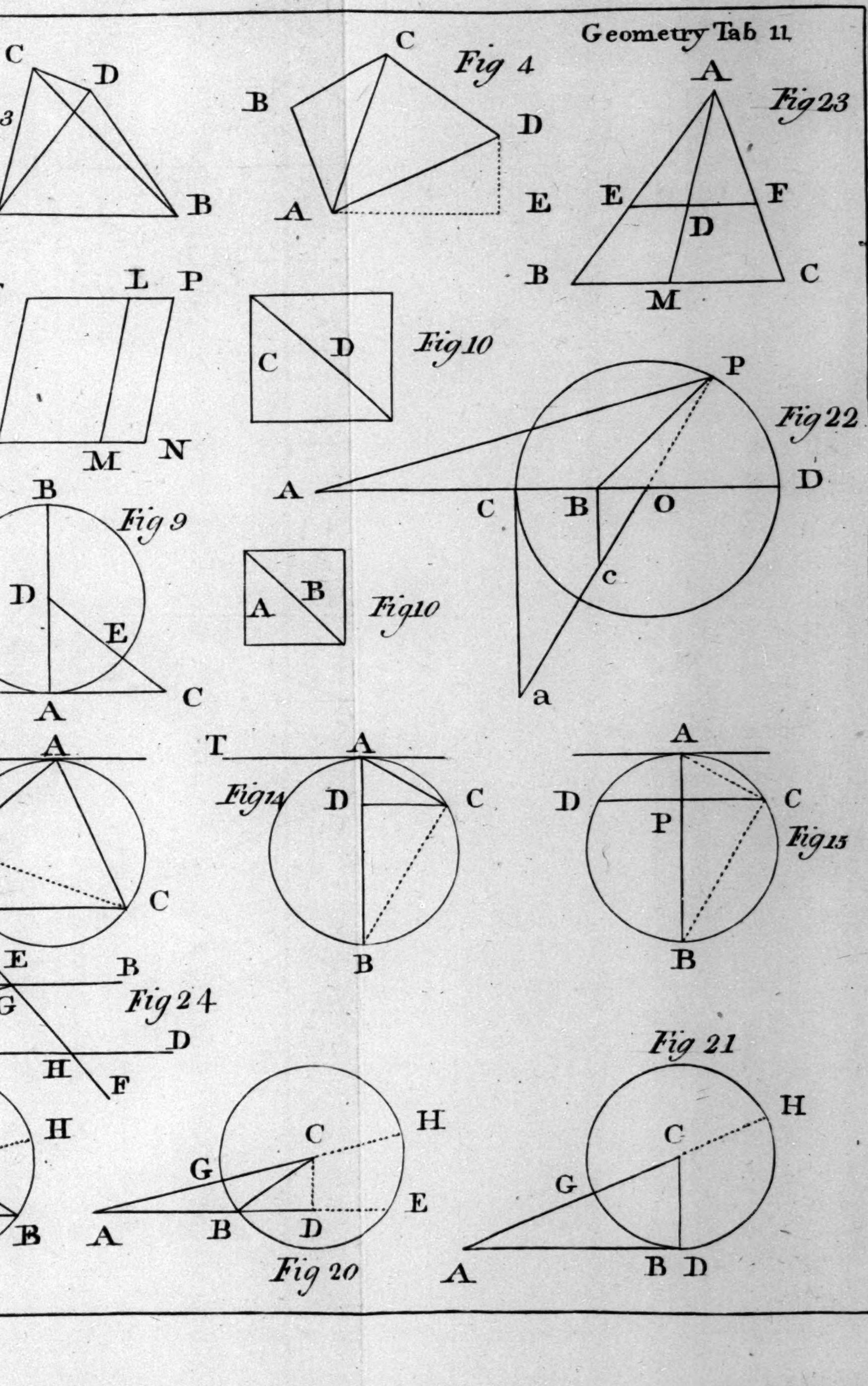


Fig 1

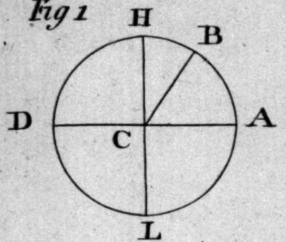


Fig 2

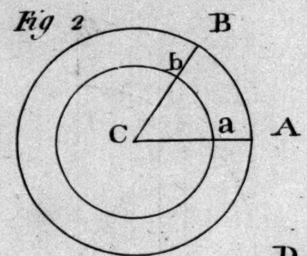


Fig 3

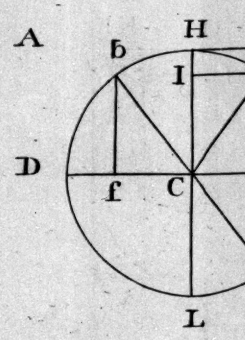


Fig 5

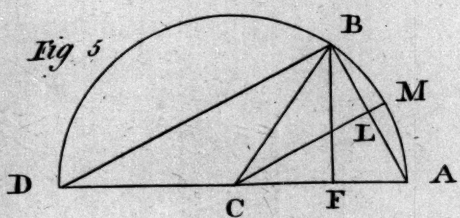
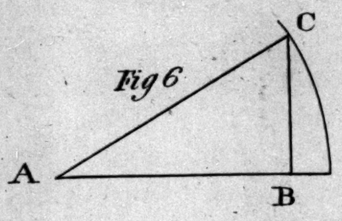


Fig 6



Fig

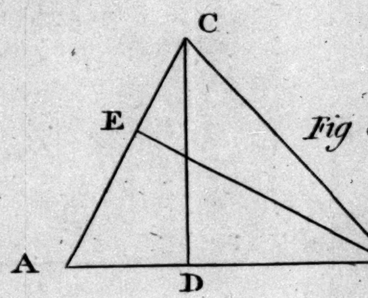


Fig 7

Fig 10

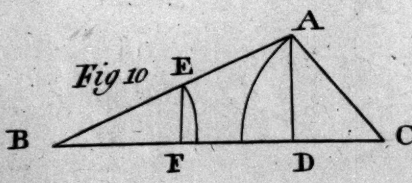


Fig 11

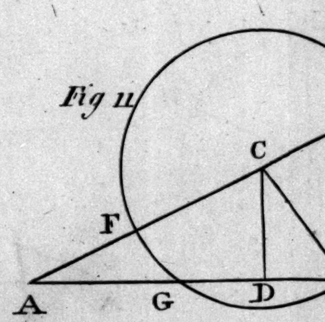


Fig 3

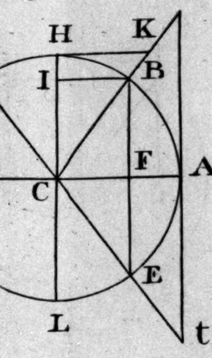


Fig 4

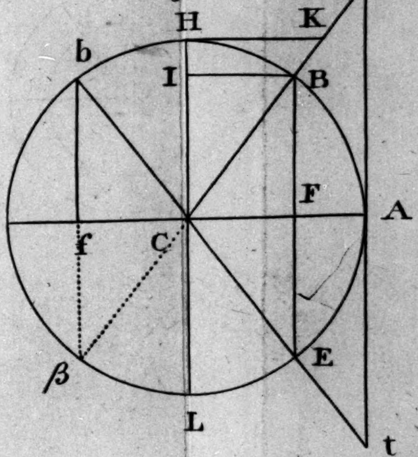


Fig 8

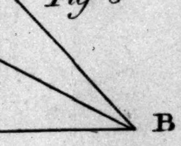
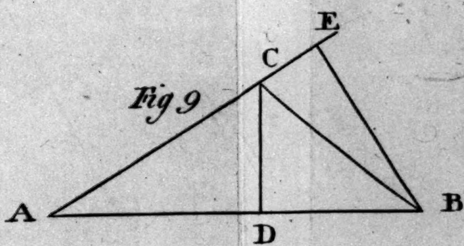


Fig 9



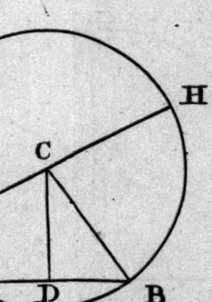


Fig 12